Chapter 6

Pensions

6.1 The budget conditions of the pension fund

The pension fund covers three types of pensions: retirement pensions, spouse pensions and disablement pensions. For each type, premiums and pensions are calculated so that the discounted present value of forecasted contributions and costs for the type of pension in question are equated from the point of view of the pension fund. To be specific, consider the individual member j. This individual enters the pension fund at the age of a_j^F and as long as the individual is not disabled or dead he or she pays contributions to the retirement pension and premiums to the spouse pension and disablement pension. If the member is not disabled and has not died prior to the retirement age a_j^R the member retires and receives retirement pension until death. If the member becomes disabled before the retirement age a disablement pension is initiated. This disablement pension runs until the death of the member and is financed by premiums paid prior to disablement. Furthermore, both disabled and non-disabled (i.e. active) members pay premiums throughout the entire life to a spouse pension, which is initiated in case of the members death, and which runs until the death of the surviving spouse. It is assumed that no individuals are older than a^L years.

The following subsections state the "budget conditions" that should hold for the premiums and for each type of pension to be actuarially fair. In stating these budget conditions we let $F_a[\cdot]$ denote the forecast operator giving the forecast made by the pension fund at age a.[1]

In the presentation below we shall focus on a representative generation, i.e. individuals of the

[1] We avoid the use of $E[\cdot]$ which may be confused with the (mathematical) expectation operator because, as shown in the next section, forecasts are deliberately made to be different from actual expectations.

same age. In DREAM there are individuals of different generations in each time period and therefore explicit reference has to made both with respect to time and age. When focusing only on a representative generation in the following, however, we shall suppress the index of time, since there is a unique linkage between time and age for any given generation. Time will be invoked only when defining the bonus, which is the same for all generations in a given time period.

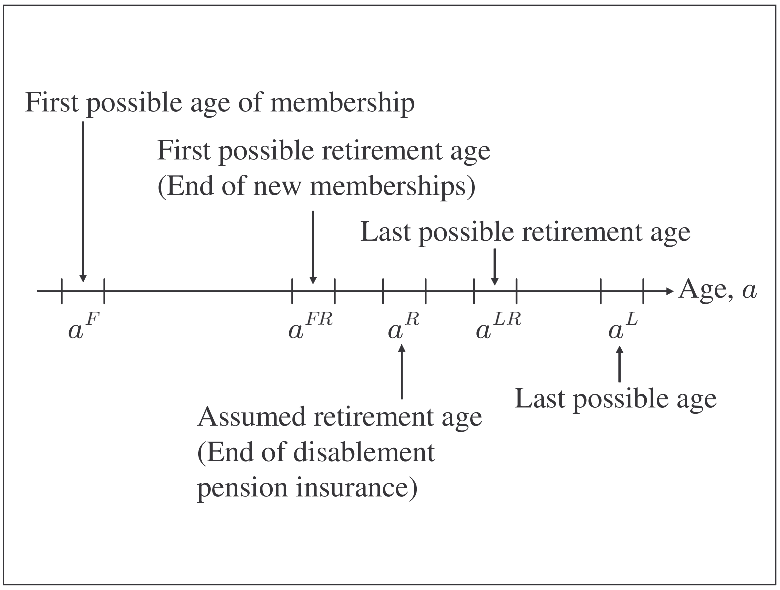

Figure 1

6.1.1 Retirement pensions

Members who are disabled receive a life-long disablement pension beginning in the period following the ocurrence of disablement. Although disablement pensioners may formally have their status changed to retirement pensioners at some age, the actual retirement pension is just a continuation of the disablement pension. We shall therefore refer to this type of members as disablement pensioners regardless of age and we shall restrict the use of the term *retirement pension* to non-disabled members who retire due to age.

The following two subsections state the budget conditions associated with retirement pension for both retired and non-retired members. The reason why a budget condition is needed for non-retired members, for whom no *actual* pension needs to be calculated, is that the pension fund each period has to calculate a forecasted retirement pension, the so-called *pension undertaking*, which forms the basis for calculating disablement and spouse pensions and various insurance premiums.

The actual retirement age of member j, a_j^R, is the decision of the individual member but retirement must take place at age a^{FR} at the earliest and all members must be retired at age a^{LR}. In addition, there is a formal retirement age a^R, where $a^{FR} \leq a^R \leq a^{LR}$. a^R is used by the pension fund to forecast retirement age. In addition the formal retirement age is important because members of age a^R and older are no longer covered by the disablement pension insurance, even if they are not yet retired. Because of this, no disablement pension premiums are paid after the age of $a^R - 1$, and when stating the budget conditions a distinction therefore has to be made with regard to whether the member is younger than a^R or not.

Non-retired members

Consider member j who is a years old, where $a_j^F \leq a < \min\left[a_j^R, a^R\right]$ that is, the member is not retired and has not passed the formal retirement age. The budget condition for this member states that:

$$
\begin{aligned}
& a_j^F \leq a < \min\left[a_j^R, a^R\right] : \\
& A_{j,a}^{B,N} + \tilde{A}_{j,a}^{B,N} + F_a \left[\sum_{y=a}^{a_j^R - 1} \left(q_{j,y} - q_{j,y}^D - q_{j,y}^{S,N} \right) \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right] \\
= \; & F_a \left[\tilde{b}_{j,a_j^R}^R \prod_{\tau=a}^{a_j^R} \left(1 + i_{t(\tau)}^Z\right)^{-1} + \sum_{y=a_j^R}^{a^L} \left(b_{j,y}^R + q_{j,y}^{S,R} \right) \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}^Z\right)^{-1} \right]
\end{aligned}
\tag{6.1}
$$

where

$A_{j,a}^{B,N}$	is beginning-of-period (exclusive of interest) assets related to the annual retirement pension for the non-retired invidual j at age a.
$\tilde{A}_{j,a}^{B,N}$	is beginning-of-period (exclusive of interest) assets related to the one-time initial retirement pension for the non-retired invidual j at age a.
$q_{j,y}$	is the gross contribution (premium) paid by member j at age y.
$q_{j,y}^D$	is the disablement pension premium paid by member j at age y.
$q_{j,y}^{S,N}$	is the spouse pension premium paid by the non-retired member j at age y.
$i_{t(\tau)}$	is the after-tax interest rate in the time period where the member is τ years old.
$\tilde{b}_{j,a_j^R}^R$	is the one-time initial retirement pension, which may be chosen by member j at the beginning of retirement, i.e. at age a_j^R
$b_{j,y}^R$	is the annual retirement pension to member j at age y.
$q_{j,y}^{S,R}$	is the spouse pension premium paid by the retired member j at age y.

The left hand side of eq. (6.1) consists of accumulated retirement pension assets plus the discounted sum of forecasted present and future *net* retirement pension contributions for the individual in question. Annual net contributions are given as the gross contributions, $q_{j,y}$, minus the insurance premiums to disablement and spouse pensions, $q^D_{j,y} + q^{S,N}_{j,y}$.

The first term on the right hand side of eq. (6.1) is the forecasted present value of the one-time pension, $\tilde{b}^R_{j,a^R_j}$, which the member may choose to recieve at the time of retirement. The second term on the right hand side consists of the discounted sum of forecasted retirement pensions and spouse pension premiums, $b^R_{j,y} + q^{S,R}_{j,y}$, made in the periods in which the member in question is retired. The premiums to the spouse pension appear as payments from the pension fund to the member after retirement because retired members do not directly pay for the spouse pension insurance. Since retirement pensioners are still covered by the spouse pension insurance, the premium is implicitly paid by the pension fund and may therefore be considered as part of a "gross" retirement pension from the fund.[2]

Equation (6.1) therefore requires individual accumulated retirement pension assets plus forecasted future net contributions to be equal to forecasted future gross retirement pension receipts.

For the non-retired member of age a where $a^R \leq a < a^R_j$ there is no disablement pension coverage, and thus the budget condition thus reads:

$$
\begin{aligned}
& a^R \leq y < a^R_j : \\
& A^{B,N}_{j,a} + \tilde{A}^{B,N}_{j,a} + F_a \left[\sum_{y=a}^{a^R_j - 1} \left(q_{j,y} - q^{S,N}_{j,y} \right) \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right] \\
= \; & F_a \left[\tilde{b}^R_{j,a^P_j} \prod_{\tau=a}^{a^R_j} \left(1 + i^Z_{t(\tau)}\right)^{-1} + \sum_{y=a^P_j}^{a^L} \left(b^R_{j,y} + q^{S,R}_{j,y} \right) \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right]
\end{aligned}
\tag{6.2}
$$

Which is the same as eq. (6.1) except that no disablement pension premiums are subtracted from gross contributions on the left hand side.

In additon to eqs. (6.1) and (6.2) we have a specific budget constraint for one-time intial

[2] Of course no spouse pension premium is paid at the last possoble age, a^L, but for ease of exposition $q^{S,R}_{j,a}$ is included at all ages. When calculating $q^{S,R}_{j,a}$ below it is set equal to zero at age a^L.

retirement pension, which reads:

$$a_j^F \leq y < a_j^R :$$
$$\tilde{A}_{j,a}^{B,N} + F_a \left[\sum_{y=a}^{a_j^R - 1} \tilde{q}_{j,y} \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right] = F_a \left[\tilde{b}_{j,a_j^P}^R \prod_{\tau=a}^{a_j^R} \left(1 + i_{t(\tau)}\right)^{-1} \right] \tag{6.3}$$

where $\tilde{q}_{j,y}$, which is part of gross contributions, $q_{j,y}$, is the specific payment to the one-time initial retirement pension. This condition just states that accumulated assets associated with the one-time pension plus the value of forecasted present and future contributions (the left hand side) should equal the forecasted present value of said pension.

It should be noted that assets and contributions associated with the one-time retirement pension are included in the over-all budget conditions of eqs. (6.1) and (6.2) while at the same time there is a specific budget constraint for the one-time pension. This is because eqs. (6.1) and (6.2) are later to be used for calculating the *pension undertaking*, which by definition is calculated using total retirement pension assets, $A_{j,a}^{B,N} + \tilde{A}_{j,a}^{B,N}$, i.e. the pension undertaking is calculated as if there will be no one-time retirement pension.

Retired members

For the just retired non-disabled individual j of age a, we have two budget conditions:

$$a = a_j^R : \tilde{d}_{j,a}^R \tilde{A}_{j,a}^{B,R} = \tilde{b}_{j,a}^R \left(1 + i_{t(a)}\right)^{-1} \tag{6.4}$$

and

$$a = a_j^R :$$
$$A_{j,a}^{B,R} + \left(1 - \tilde{d}_{j,a}^R\right) \tilde{A}_{j,a}^{B,R} = F_a \left[\sum_{y=a}^{a^L} \left(b_{j,y}^R + q_{j,y}^{S,R}\right) \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right] \tag{6.5}$$

where, in addition to variables already defined:

$\tilde{d}_{j,a}^R$	is a variable equal to 1 if member j chooses to recieve a one-time initial retirement pension at age a.
$\tilde{A}_{j,a}^{B,R}$	is beginning-of-period (exclusive of interest) assets related to the one-time retirement pension for the retired invidual j at age a.
$A_{j,a}^{B,R}$	is beginning-of-period (exclusive of interest) assets related to the annual retirement pension for the retired invidual j at age a.

Eq. (6.4) states that the discounted value of the one-time pension (the right hand side) should equal accumulated assets related to the one-time pension, if the member chooses said pension, i.e. if $\tilde{d}_j^R = 1$. The discounting on the right hand side is due to the fact that $\tilde{A}_{j,a}^{B,R}$ on the left hand side is measured exclusive of interest at age a.

Eq. (6.5) states the budget constraint associated with the annual retirement pension. The left hand side is given by accumulated assets, which include one-time retirement pension assets if the member has chosen not to receive the one-time pension. The right hand side consists of the discounted present and forecasted future expenses associated with the annual retirement pension. These expenses are given by the retirement pension itself and by premiums to the spouse pension insurance.

It should be noted that assets for a *retired* member, i.e. $A_{j,a}^{B,R}$ and $\tilde{A}_{j,a}^{B,R}$ have a superscript R whereas assets for the *non-retired* individual in eqs. (6.1), (6.2) and (6.3), i.e. $A_{j,a}^{B,N}$ and $\tilde{A}_{j,a}^{B,N}$, have a superscript N. For a given individual $A_{j,a}^{B,R}$ and $\tilde{A}_{j,a}^{B,R}$ are of course strictly related to (values at previouos ages of) $A_{j,a}^{B,N}$ and $\tilde{A}_{j,a}^{B,N}$ but the distinction has to be made because DREAM includes only *average* pensions instead of individual pensions. Since members do not all retire at the same age we need to be able do distinguish between assets for retired and non-retired members.

For a continued annual retirement pension, i.e. for a member of age a where $a_j^R < a \leq a^L$ we have

$$a_j^R < a \leq a^L : A_{j,a}^{B,R} = F_a \left[\sum_{y=a}^{a^L} \left(b_{j,y}^R + q_{j,y}^{S,R} \right) \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right] \tag{6.6}$$

This condition is similar to eq. (6.5) except that assets associated with the one-time pension are no longer relevant.

6.1.2 Spouse pensions

This section presents the principles (budget constraints) behind the calculation of spouse pension insurance premiums and the actual spouse pensions. Spouse pensions are given by a life-long annual pension and (potentially) by a one-time amount, which initiates the spouse pension.

When initiated, the annual spouse pension is defined in terms of either the actual pension of the deceased member (in case the member was either retired or disabled) or, if the deceased member was non-retired, in terms of the pension undertaking. As a result, we need to state budget conditions for spouse pensions associated with all three types of members, and in case the member is either disabled or retired we even have to keep track of when disablement or retirement took place, because pensions will be depending on this. However, for all three types of members, the basic principles behind the calculations are exactly the same, and except for the presentation of notation, the reader is therefore adviced to focus on only one of the following three sub-sections.

Spouses of non-retired members

Premiums Consider non-retired member j who is a years old where $a_j^F \leq a < a_j^R$. In case member j dies before turning $a+1$ years old and in case there is a living spouse, a spouse pension for the spouse called[3] $\hat{\jmath}$ is initiated at the age[4] $y+1$ and this pension runs until the death of the spouse. For the spouse pension premium paid by member j at age age a to be actuarially fair it must be the case that

$$a_j^F \leq a < a_j^R :$$
$$q_{j,a}^{S,N} = F_a \left[\tilde{b}_{\hat{\jmath},a+1}^{S,N} \left(1 + i_{t(a+1)}\right)^{-1} + \sum_{y=a+1}^{a^L} b_{\hat{\jmath},y,a+1}^{S,N} \prod_{\tau=a+1}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right] \tag{6.7}$$

where, in addition to variable already defined,

- $q_{j,a}^{S,N}$ is the spouse pension premium paid at age a by the non-retired member j.
- $\tilde{b}_{\hat{\jmath},a+1}^{S,N}$ is the one-time initial spouse pension, which (together with the life-long annual pension) the spouse pension to the $a+1$ year old widow/widower $\hat{\jmath}$ whose late spouse was non-retired member j.
- $b_{\hat{\jmath},y,s}^{S,N}$ is the at age s initiated life-long annual spouse pension to the y year old spouse $\hat{\jmath}$ who is the widow/widower after the non-retired member j.

The left hand side of eq. (6.7) is the spouse pension premium paid by member j to cover the costs of the spouse pension which is initiated in case the member does not live to become $a+1$ years old.

The right hand side of eq. (6.7) is the forecasted value of the total present value cost associated with this spouse pension. The cost consists of the sum of the one-time pension and the life-long annual pension.

[3] Depending on the context we use $\hat{\jmath}$ either to denote the spouse of member j or $\hat{\jmath}$ to denote the late member associated with spouse j.

[4] It is assumed that both spouses are of the same age.

For the premium to be actuarially fair it should be equal to the forecasted costs. Since death may occur at all ages, eq (6.7) must hold for each age a, $a_j^F \leq a < a_j^R$, at which member j is alive and not retired (or disabled).

It should be noted that the calculation of the premium follows the normal principle of insurance in that the premium paid by each member in each period is meant to cover the *entire* forecasted cost associated with the spouse pension in case of the member's death before the following period. If the member does not die the money is 'lost' and in the following period a new premium will be paid in order to cover the forecasted cost associated with the member dying before the next period.

Pensions Now consider the individual j who is a years old and experiences his or her *first* period as a spouse pensioner after a non-retired member. In the calculation of the spouse pension the budget condition requires that:

$$a_{\hat{j}}^F < a \leq a_{\hat{j}}^R :$$
$$A_{j,a,a}^{B,S,N} = F_a \left[\tilde{b}_{j,a}^{S,N} \left(1 + i_{t(a)}\right)^{-1} + \sum_{y=a}^{a^L} b_{j,y,a}^{S,N} \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right] \tag{6.8}$$

where, in addition to variables already defined:

$A_{j,a,s}^{B,S,N}$ is beginning-of-period (exclusive of interest) assets related to the spouse pension for individual j of age a who first received spouse pension at age s and whose late spouse was a non-retired member.

Eq. (6.8) requires that assets (the left hand side) equal the forecasted value of total discounted present and future costs associated with the spouse pension. The costs consist of the one-time initial pension (the first term on the right hand side) and expences to the life-long annual spouse pension (the second term on the right-hand-side).

For the *continued* spouse pension, i.e. for a spouse pensioner who is a years old and experienced the first period as a spouse pensioner at age $s < a$ the budget condition reads:

$$a_{\hat{j}}^F + 1 < y \leq a^L :$$
$$A_{j,a,s}^{B,S,N} = F_a \left[\sum_{y=a}^{a^L} b_{j,y,s}^{S,N} \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right], \quad a > s \tag{6.9}$$

Eq. (6.9) is identical to eq. (6.8) except for the fact that there is no one-time initial spouse pension.

For a given spouse pensioner eq. (6.8) or (6.9) should hold at each age at which the pensioner is alive.

Spouses of retired members

Premiums Consider retired member j who is a years old where $a_j^R \leq a < a^L$. For the spouse pension premium paid by member j age age a to be actuarially fair it must be the case that

$$a_j^R \leq a < a^L :$$
$$q_{j,a}^{S,R} = F_a \left[\tilde{b}_{\hat{\jmath},a+1}^{S,R} \left(1 + i_{t(a+1)}\right)^{-1} + \sum_{y=a+1}^{a^L} b_{\hat{\jmath},y,a+1}^{S,R} \prod_{\tau=a+1}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right] \tag{6.10}$$

where

$q_{j,a}^{S,R}$ is the spouse pension premium paid at age a by the retired member j.

$\tilde{b}_{\hat{\jmath},a+1}^{S,R}$ is the one-time initial spouse pension, which (together with the life-long annual pension) i the spouse pension to the $a+1$ year old widow/widower $\hat{\jmath}$ whose late spouse was retired member j.

$b_{\hat{\jmath},y,s}^{S,R}$ is the at age s initiated life-long annual spouse pension to the y year old spouse $\hat{\jmath}$ who is the widow/widower after the retired member j.

Just as eq. (6.7) above eq. (6.10) requires the premium to be equal to the forecasted total cost associated with the spouse pension in case of the member's death. Eq (6.10) should hold for each age a, $a_j^R \leq a < a^L$, at which member j is alive.

Pensions In the calculation of the spouse pension individual j who is a years old and experiences his or her *first* period as a spouse pensioner after a retired member the budget condition requires that:

$$a_{\hat{\jmath}}^R < a \leq a^L :$$
$$A_{j,a,a}^{B,S,R} = F_a \left[\tilde{b}_{j,a}^{S,R} \left(1 + i_{t(a)}\right)^{-1} + \sum_{y=a}^{a^L} b_{j,y,a}^{S,R} \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right] \tag{6.11}$$

where

$A_{j,a,s}^{B,S,R}$ is beginning-of-period (exclusive of interest) assets related to the spouse pension for individual j of age a who first received spouse pension at age s and whose late spouse was a retired member.

Eq. (6.11) is analogous to eq. (6.8) in that it requires assets (the left hand side) to equal the forecasted value of discounted present and forecasted future costs associated with the spouse pension.

For the *continued* spouse pension, i.e. for a spouse pensioner who is a years old and experienced the first period as a spouse pensioner at age $s < a$ the budget condition reads:

$$
\begin{aligned}
& a_{\hat{\jmath}}^{R} + 1 < y \le a^{L}: \\
& A_{j,a,s}^{B,S,R} = F_a \left[\sum_{y=a}^{a^L} b_{\hat{\jmath},y,s}^{S,R} \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right], \quad a > s
\end{aligned}
\tag{6.12}
$$

Eq. (6.12) is identical to eq. (6.11) except for the fact that there is no one-time pension.

For a given spouse pensioner eq. (6.11) or (6.12) should hold at each age at which the pensioner is alive.

Spouses of disabled members

Premiums Disabled member j who experienced his or her first period as a disablement pensioner at age d and who is now $a \ge d$ years old pays a spouse pension premium equal to:

$$
\begin{aligned}
& a^{F} < a < a^{L}: \\
& q_{j,a,d}^{S,D} = F_a \left[\tilde{b}_{\hat{\jmath},a+1}^{S,D} \left(1 + i_{t(a+1)}\right)^{-1} + \sum_{y=a+1}^{a^L} b_{\hat{\jmath},y,a+1,d}^{S,D} \prod_{\tau=a+1}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right]
\end{aligned}
\tag{6.13}
$$

where

$q_{j,a,d}^{S,D}$ is the spouse pension premium paid at age a by the disabled member j who first received disablement pension at age d.

$\tilde{b}_{\hat{\jmath},a+1}^{S,D}$ is the one-time initial spouse pension, which (together with the life-long annual pension) initiates the spouse pension to the $a+1$ year old widow/widower $\hat{\jmath}$ whose late spouse was disabled member j.

$b_{\hat{\jmath},y,s,d}^{S,D}$ is the at age s initiated life-long annual spouse pension to the y year old spouse $\hat{\jmath}$ who is the widow/widower after the disabled member j who first received disablement pension at age d.

Eq. (6.13) thus states that the premium should cover the forecasted value of discounted future costs associated with the spouse pension. Eq. (6.13) should hold for each age a at which member j is alive.

Pensions For the a (where $a_{\hat{j}}^{ZF,First}+1<y\leq a^{ZF,Last}$)[5] year old widow/widower after a disabled member the first period spouse pension budget constraint reads:

$$a_{\hat{j}}^{ZF,First}+1<y\leq a^{ZF,Last}:$$
$$A_{j,a,a,d}^{B,S,D}=F_a\left[\tilde{b}_{j,a}^{S,D}\left(1+i_y^Z\right)^{-1}+\sum_{y=a}^{a^L}b_{j,y,a,d}^{S,D}\prod_{\tau=a}^{y}\left(1+i_{t(\tau)}\right)^{-1}\right] \tag{6.14}$$

where

$A_{j,a,s,d}^{B,S,D}$ is beginning-of-period (exclusive of interest) assets related to the spouse pension for individual j of age a who first received spouse pension at age s and whose late spouse was a disabled member who first received diablement pension at age d.

Just like eq. (6.8) above eq. (6.14) requires assets to equal the forecasted value of the discounted sum of present and forecasted future costs associated with the spouse pension. This is also the case for the budget constraint for an continued spouse pension, i.e. for a pension to the a year old spouse who was widowed at age $s<a$, with the exception that in the latter case there is no one-time pension. This condition thus reads:

$$a_{\hat{j}}^{ZF,First}+2<y\leq a^{ZF,Last}:$$
$$A_{j,a,s,d}^{B,S,D}=F_a\left[\sum_{y=a}^{a^L}b_{j,y,s,d}^{S,D}\prod_{\tau=a}^{y}\left(1+i_{t(\tau)}\right)^{-1}\right],\quad y>s \tag{6.15}$$

For a given spouse pensioner eq. (6.14) or (6.15) should hold at each age at which the pensioner is alive.

6.1.3 Disablement pensions

****

As mentioned above non-retired members are covered by the disablement pension insurance until the age of a^R. The budget constraints associated with premiums and pensions follow the same principles as for spouse pensions and are given in the next two subsections.

****

[5] The youngest spouse pensioners after disabled members are those of age A^0+2 whose spouses were disabled at age A^0 and died at age A^0+1.

Premiums In addition to the spouse pension premium the non-retired member j who is a years old (where $a_j^F \leq a < \min\left[a_j^R, a^R\right]$) also pays a premium to cover the disablement pension, which is initiated in case the member is disabled and thus experiences the first period as a disablement pensioner at age $a+1$. For the premium to be fair it should be given by:

$$a_j^F \leq a < \min\left[a_j^R, a^R\right] :$$
$$q_{j,a}^D = F_a \left[\tilde{b}_{j,a+1}^D \left(1 + i_{t(a+1)}\right)^{-1} + \sum_{y=a+1}^{a^L} \left(b_{j,y,a+1}^D + q_{j,y,a+1}^{S,D}\right) \prod_{\tau=a+1}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right] \quad (6.16)$$

where

$q_{j,a}^D$	is the disablement pension premium paid by member j at age a.
$\tilde{b}_{j,a+1}^D$	is the one-time initial disablement pension, which (together with the life-long annual pension) ini the disablement pension to the $a+1$ year old member j.
$b_{j,y,d}^D$	is the at age d initiated life-long annual disablement pension to the y year old disabled member j.
$q_{j,y,d}^{S,D}$	is the spouse pension premium paid at age y by the disabled member j who first received disablement pension at age d.

The left hand side of eq. (6.16) is the premium paid at age a to cover the disablement pension in case the member becomes disabled before turning $a+1$ years old and thus receives a disablement pension beginning af age $a+1$. The right hand side of eq. (6.16) is the forecasted value of the discounted sum of future disablement pensions and premiums to the spouse pension insurance. The disablement pension consists of a one-time pension received in the first period as a disablement pensioner (the first term on the right hand side) as well as of a life-long annual pension included in the second term on the right hand side. The second term also includes the premiums to the spouse pension insurance. These premiums appear as payment from the pension fund to the member after disablement, because disabled persons do not directly pay spouse pension premiums. Since disablement pensioners are still covered by the spouse pension insurance, the premium is implicitly paid by the pension fund and may therefore be considered as part of a "gross" disablement pension from the fund.

Pensions For the a year old individual experiencing his or her first period as a disablement pensioner (where $a_j^F < a \leq \min\left[a_j^R, a^R\right]$) the budget condition is

$$a_j^F < a \leq \min\left[a_j^R, a^R\right]$$
$$A_{j,a,a}^{B,D} = F_a \left[\tilde{b}_{j,a}^D \left(1 + i_{t(a)}\right)^{-1} + \sum_{y=a}^{a^L} \left(b_{j,y,a}^D + q_{j,y,a}^{S,D}\right) \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right] \quad (6.17)$$

where

$A^{B,D}_{j,a,d}$ is beginning-of-period (exclusive of interest) assets related to the disablement pension for member j of age a who first received disablement pension at age d.

Eq. (6.17) requires assets to equal the forecasted value of present and discounted future cost associated with the disablement pension. Costs include a one-time initial disablement pension (the first term on the right hand side) as well as the life-long annual disablement pension included in the second term. In addition, since a disabled member does not pay for the spouse pension insurance but is still covered by the insurance, the spouse pension premium is paid by the pension fund and is therefore included in the costs associated with the disablement pension in the second term on the right hand side.

For an continued disablement pension the one-time initial pension is no longer part of the cost, so for the disabled member of age a who experienced disablement first at age $d < a$ (where $a^F_j < d \leq \min\left[a^R_j, a^R\right]$) the budget constraint is:

$$a^F_j + 1 < y \leq a^L :$$
$$A^{B,D}_{j,a,d} = F_a \left[\sum_{y=a}^{a^L} \left(b^D_{j,y,d} + q^{S,D}_{j,y,d} \right) \prod_{\tau=a}^{y} \left(1 + i_{t(\tau)}\right)^{-1} \right], \quad a > d \tag{6.18}$$

For a given disablement pensioner eq. (6.17) or (6.18) should hold at each age at which the pensioner is alive.

6.2 Forecasting, pension undertakings and principles of precaution

6.2.1 Actuarial constants

The preceding section presents the various budget constraints required to hold in order for pensions and premiums to be actuarially fair. All these constraints require the pension fund to forecast, among other things, future events of death or disablement. In doing this the pension fund uses the following probabilities, which, as mentioned below, are chosen to be moderate in relation to actual mortality, disablement and marriage rates.

$\bar{r}^M_{x,t}$	Probability used in period t to forecast the event that an $x-1$ year old member dies before turning x years old.
$\bar{r}^D_{x,t}$	Probability used in period t to forecast the event that an $x-1$ year old non-retired member becomes disabled before turning x years old.
$\bar{r}^S_{x,t}$	Probability used in period t to forecast the event that for an $x-1$ year old member there will be a living spouse in the following year (i.e. a spouse of age x).

It should be noted that these probabilities depend on both age and time, but not on gender because according to rules stated by Danish authorities discrimination based on gender is not allowed.[6]

The pension fund also needs to forecast the future value of the (after-tax) interest rate, which is done by using the so-called *base interest rate*, $\bar{\imath}_t$. $\bar{\imath}_t$ is thus the forecast used in period t of the actual after-tax interest rate of all future periods.

The probabilities and the base interest rate are part of the so-called technical basis[7] of the pension fund. Because the technical basis is used to calculate actual pensions and to make promises about future pensions, authorities require the technical basis to be such as to avoid making unrealistically high promises. However, since actual probabilities and interest rates may change over time, this may require the technical basis to also change over time, which is why the probabilities and the base interest rate are indexed according to the period t in which they are used. In should be noted however, that the technical basis is by no means changed each period.

Based on the above probabilities we may now define the following actuarial constants

$$l_{a,t} \equiv \prod_{x=1}^{a} \left(1-\bar{r}^M_{x,t}\right)$$
$$l^A_{a,t} \equiv \prod_{x=1}^{a} \left(1-\bar{r}^D_{x,t}\right)$$

$l_{a,t}$ is the forecasted (in period t) probability of survival from time of birth to the age of a and $l^A_{a,t}$ is the probability that a member is not disabled from time of birth to the age of a (provided that the person has not died).[8] Based on these constants and the base interest rate

[6] However, different pension funds may be allowed to use different sets of probabilities. This feature is not present in the model since the labour market pension system in DREAM is modelled as a unique pension fund.

[7] In Danish: 'Det tekniske grundlag' or 'Beregningsgrundlaget'.

[8] The probability that a person survives from time of birth until the age of a *and* is not disabled during that period is $l_{a,t}l^A_{a,t}$.

we now define

$$
\begin{aligned}
D_{a,t} &\equiv l_{a,t}\left(\frac{1}{1+\bar{\imath}_t}\right)^a \\
D^A_{a,t} &\equiv l^A_{a,t} l_{a,t}\left(\frac{1}{1+\bar{\imath}_t}\right)^a
\end{aligned}
$$

$D_{a,t}$ and $D^A_{a,t}$ are finally used to calculate the following two fractions (where $a \leq y$):

$$
\begin{aligned}
\frac{D_{y,t}}{D_{a,t}} &= \frac{l_{y,t}}{l_{a,t}}\left(\frac{1}{1+\bar{\imath}_t}\right)^{y-a} \\
&= \prod_{x=a+1}^{y}\left(1-\bar{r}^M_{x,t}\right)\left(\frac{1}{1+\bar{\imath}_t}\right)^{y-a}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{D^A_{y,t}}{D^A_{a,t}} &= \frac{l^A_{y,t}}{l^A_{a,t}}\frac{l_{y,t}}{l_{a,t}}\left(\frac{1}{1+\bar{\imath}_t}\right)^{y-a} \\
&= \prod_{x=a+1}^{y}\left(1-\bar{r}^D_{x,t}\right)\prod_{x=a+1}^{y}\left(1-\bar{r}^M_{x,t}\right)\left(\frac{1}{1+\bar{\imath}_t}\right)^{y-a}
\end{aligned}
$$

$\frac{D_{y,t}}{D_{a,t}}$ denotes the forecasted (in period t) discounted (using the base interest rate) cost of promising a member of age a one monetary unit at age y conditional on the person being alive at age y.

$\frac{D^A_{y,t}}{D^A_{a,t}}$ denotes the forecasted (in period t) discounted (using the base interest rate) cost of promising an active member of age a one monetary unit at age y conditional on the person being alive and active at age y.[9]

For later use we note that

$$
\frac{D_{y,t}}{D_{a-1,t}} = \frac{1-\bar{r}^M_{a,t}}{1+\bar{\imath}_t}\frac{D_{y,t}}{D_{a,t}} \Leftrightarrow \tag{6.19}
$$

$$
\frac{D_{y,t}}{D_{a,t}} = \frac{1+\bar{\imath}_t}{1-\bar{r}^M_{a,t}}\frac{D_{y,t}}{D_{a-1,t}} \tag{6.20}
$$

and

$$
\frac{D^A_{y,t}}{D^A_{a-1,t}} = \frac{\left(1-\bar{r}^D_{a,t}\right)\left(1-\bar{r}^M_{a,t}\right)}{1+\bar{\imath}_t}\frac{D^A_{y,t}}{D^A_{a,t}} \Leftrightarrow \tag{6.21}
$$

$$
\frac{D^A_{y,t}}{D^A_{a,t}} = \frac{1+\bar{\imath}_t}{\left(1-\bar{r}^D_{a,t}\right)\left(1-\bar{r}^M_{a,t}\right)}\frac{D^A_{y,t}}{D^A_{a-1,t}} \tag{6.22}
$$

[9] The interpretation of $\frac{D_{y,t}}{D_{a,t}}$ and $\frac{D^A_{y,t}}{D^A_{a,t}}$ also shows that although $l_{a,t}$ and $l^A_{a,t}$ denote probabilities starting at birth, which may seem a bit odd, they are only used in expressions, which are based on the fact that an individual has already survived until age a and for all practical purposes a will be at least a^F_j - the age at which the member has entered the pension fund.

6.2.2 Pension undertakings

When initiating a spouse pension in period t to the spouse of a deceased member who was either disabled or retired, the initial spouse pension is given by w_τ^S times the deceased member's disablement or retirement pension of the previous period.[10] The question remains, however, of how to *initiate* disablement pensions to disabled members and spouse pensions to spouses who have been married to non-retired members because in both these cases there is no pension of the previous period which can form the basis of disablement or spouse pension. These problems are solved by the definition of the pension undertakings, $b_{j,a}^U$, which are the basis for the calculation of the various pension types that are offered by the pension fund. The first year's receipts of disablement pension and of spouse pensions for widows/widowers after non-retired members are thus expressed in terms of the pension undertakings and therefore all information relevant for the calculation of a specific pension in case of an event that releases the pension is summarized in the pension undertaking.[11] The pension undertakings are calculated at every age, a, for each active member until the year before retirement, a_j^R.

6.2.3 Principles of precaution

Both the calculation of the pension undertakings and the pensions and premiums introduced above in section 6.1 require that future pension undertakings, future premiums and future pensions are forecasted. These forecasts are made given certain *principles of precaution* the aim of which is to ensure .... The principles of precaution consist of:

- **Low interest rate.** In its forecasts the pension fund uses the base interest rate, $\bar{\imath}$, which is lower than the (expected) after-tax interest rate i. This is done to reduce the risk of promising too high future returns. Each year a correction is made to take account of the fact that the achieved interest rate after tax in the specific year deviate from the base

[10] Furthermore spouse pensions are assumed (in period t) to run for the entire remaining life of the spouse but to decrease at the rate g_t^S over time, which is meant to capture the fact that real spouse pensions consist of two types:

- Pensions which run for entire remaining life of the spouse.
- Pensions which run for a limited number of years.

[11] For spouse pensioners of this category the initial value of the life-long spouse pension is given by w_t^S times the deceased member's pension undertaking of the previous period and initial life-long disablement pensions are given by w_t^D times the pension undertaking of the previous period.

interest rate.

- **Moderate probabilities of death, disablement and marriage**. In forecasting the probabilities of death, disablement and the presence of a spouse to receive spouse pension in case of a members death the pension fund uses probabilities that prevent promising too high future pensions. This requires the probabilities of death to be lower than what is actually expected and probabilities of marriage and disablement to be higher.[12] Each year a correction in the form of a bonus on accumulated assets for all members and spouse pensioners is made to take account of the fact that actual patterns of death, disablement and the presence of a spouse are different from what has been forecasted.

- **Constant future contributions**. The pension fund assumes the future contributions for each non-retired member to remain at the present level. Each year a correction of the pension undertaking is made to take account of the fact that the achieved contributions in the specific year deviate from the contributions of the previous year.

- **Constant future pensions**. In its forecasts the pension fund assumes constant future pensions over time. Forecasts are corrected each year according to the rules stated above.

- **Constant future pension undertakings**. When making forecasts the pension fund assumes constant future pension undertakings over time. Forecasts are corrected each year according to the rules stated above.[13]

The principles of precaution regarding the probabilities, the interest rate, contributions, pension undertakings and pensions are summarised in the following table. It should be noted that terms included in *the technical basis*, i.e. probabilities, the base interest rate and the constants w_t^D, $w_t^{;S}$ and g_t^S used when calculating disablement and spouse pensions are indexed according to time t because it will later be important to keep track on the exact technical basis used when making forecasts. The one-time initial pensions related to spouse and disablement pensions, $\tilde{b}_{j,a,t}^{S,N}$, $\tilde{b}_{j,a,t}^{S,R}$, $\tilde{b}_{j,a,t}^{S,D}$and $\tilde{b}_{j,a,t}^{D}$ are not directly determined by the individual member, but are fixed amounts determined by the pension fund. These may therefore also be changed over time

[12] Actually, it requires probabilities of death used when calculating retirement, disablement and spouse pensions to be lower than what is actually expected and probabilities of death used when calculating spouse pension premiums to be higher than what is actually expected. However, we use the same set of death probabilities for all purposes.

[13] Actually this requirement is superflous in the sense that given constant future contributions and constant future pensions it must be the case that future pension undertakings are constant.

and are consequently indexed to time. For the various contributions, premiums and life-long pensions related to the *specific individual*, we shall continue to suppress explicit reference to time because this is implicitly given by the age of the individual.

The first table gives forecasts made in period t regarding period $\tau > t$ while the second table gives forecasts made at age a regarding age $y > a$. The function $t(a)$ denotes the time period in which the individual is a years old.

Variable	Forecast made at time t
i_τ	$\bar{\imath}_t$
$\bar{r}^M_{a,\tau}$	$\bar{r}^M_{a,t}$
$\bar{r}^D_{x,\tau}$	$\bar{r}^D_{x,t}$
$\bar{r}^S_{x,\tau}$	$\bar{r}^S_{x,t}$
$\tilde{b}^{S,N}_{j,a,\tau}$	$\tilde{b}^{S,N}_{j,a,t}$
$\tilde{b}^{S,R}_{j,a,\tau}$	$\tilde{b}^{S,R}_{j,a,t}$
$\tilde{b}^{S,D}_{j,a,\tau}$	$\tilde{b}^{S,D}_{j,a,t}$
$\tilde{b}^{D}_{j,a,\tau}$	$\tilde{b}^{D}_{j,a,t}$

Variable	Forecast made at age a	
$q_{j,y}$	$q_{j,a}$	
$\tilde{q}_{j,y}$	$\tilde{q}_{j,a}$	
$b_{j,y}^{ZF,U}$	$b_{j,a}^{ZF,U}$	
$b_{j,y}^{ZF,R}$	$b_{j,a}^{ZF,U}$,	$a < a_j^R$
$b_{j,y}^{ZF,R}$	$b_{j,a}^{ZF,R}$,	$a \geq a_j^R$
$b_{j,y,s}^{S,N}$	$\left(\frac{1}{1+g_{t(a)}^S}\right)^{y-s} w_{y,t(a)}^S b_{\hat{j},a}^{ZF,U}$,	$a < s$
$b_{j,y,s}^{S,N}$	$\left(\frac{1}{1+g_{t(a)}^S}\right)^{y-a} b_{j,a,s}^{S,N}$,	$a \geq s$
$b_{j,y,s}^{S,R}$	$\left(\frac{1}{1+g_{t(a)}^S}\right)^{y-s} w_{y,t(a)}^S b_{\hat{j},a}^{ZF,R}$,	$a < s$
$b_{j,y,s}^{S,R}$	$\left(\frac{1}{1+g_{t(a)}^S}\right)^{y-a} b_{j,a,s}^{S,R}$,	$a \geq s$
$b_{j,y,s,d}^{S,D}$	$\left(\frac{1}{1+g_{t(a)}^S}\right)^{y-s} w_{y,t(a)}^S b_{\hat{j},a,d}^{D}$,	$a < s$
$b_{j,y,s,d}^{S,D}$	$\left(\frac{1}{1+g_{t(a)}^S}\right)^{y-a} b_{j,a,s,d}^{S,D}$,	$a \geq s$
$b_{j,y,d}^{D}$	$w_{y,t(a)}^D b_{j,a}^U$,	$a < d$
$b_{j,y,d}^{D}$	$b_{j,a,d}^{D}$,	$a \geq d$

Note that the forecasted spouse pensions also include the fact that life-long spouse pensions are decreasing at rate g^S over time.

Finally, the pension fund needs to forecast the retirement age for non-retired member j, a_j^R. This is done in the following way

$$
\begin{aligned}
F_a\left(a_j^R\right) &= a^R \text{ for } a < a^R \\
F_a\left(a_j^R\right) &= a+1 \text{ for } a \geq a^R
\end{aligned}
$$

For non-retired members younger than the technical retirement age, a^R, it is thus assumed, that they will retire at age a^R. For non-retired members who have reached (at least) the age of a^R, it is assumed that they will retire in the following period.

6.3 Derivation of individual pensions and premiums

This section combines the budget constraints for the calculation of premiums and pensions with the forecasting procedures of the preceding section to calculate the actual pensions and premiums of all types of individual. Compared with section 6.1 the order of presentation is changed in that we begin with spouse pensions, proceed with disablement pensions before finally calculating pension undertakings and retirement pensions. This is so because spouse pension premiums need to be known in order to calucalute all other types of pensions, and disablement pension premiums need to be known in order to calculate the pension undertakings.

6.3.1 Spouse Pensions

Spouses of non-retired members

Premiums Using the principles of precaution to form the forecast in eq. (6.7) we obtain

$$q_{j,a}^{S,N} = \frac{1}{1+\bar{\imath}_{t(a)}} \bar{r}_{a+1,t(a)}^{M} \bar{r}_{a+1,t(a)}^{S} \left(\tilde{b}_{j,a+1,t(a)}^{S,N} + \sum_{y=a+1}^{a^L} \left(\frac{1}{1+g_{t(a)}^{S}} \right)^{y-(a+1)} w_{y,t(a)}^{S} b_{j,a}^{U} \frac{D_{y,t(a)}}{D_{a+1,t(a)}} \right) \tag{6.23}$$

in case $a_j^{ZF,First} \leq a < a_j^{ZF,R}$.

$\bar{r}_{a+1,t(a)}^{M} \bar{r}_{a+1,t(a)}^{S}$ is the combined probability that member j dies before turning $a+1$ years old *and* that there is a living spouse. The term in the paranthesis is the total forecasted cost of the spouse pension given that the member has died. This includes the one-time initial spouse pension, which is paid in the period after the death of the member, and it also includes the life-long annual pension, which further takes into account the fact that the spouse dies with a certain probability as time evolves and that the initital spouse pension is given by $w_{y,t(a)}^{S} b_{j,a}^{U}$ and decreases with rate $g_{t(a)}^{S}$ over time. The total forecasted cost in the paranthesis is measured in present value at age $a+1$, that is the age of the spouse when the spouse pension is initiated. Multiplication with $\frac{1}{1+\bar{\imath}_{t(a)}}$ (the first term on the right hand side) discounts this sum until the time where the member is a years old and has to pay the premium.

Defining

$$a^F \leq a < a^L : \tilde{\beta}^{S,N}_{j,a,t} \equiv \frac{1}{1+\bar{\imath}_t}\bar{r}^M_{a+1,t}\bar{r}^S_{a+1,t}\tilde{b}^{S,N}_{j,a+1,t} \tag{6.24}$$

$$a^F \leq a < a^L : \beta^S_{a,t} \equiv \frac{1}{1+\bar{\imath}_t}\bar{r}^M_{a+1,t}\bar{r}^S_{a+1,t}\sum_{y=a+1}^{a^L}\left(\frac{1}{1+g^S_t}\right)^{y-(a+1)} w^S_{a,t}\frac{D_{y,\tau}}{D_{a+1,\tau}} \tag{6.25}$$

$$a = a^L : \tilde{\beta}^{S,N}_{j,a,t} \equiv \beta^S_{a,t} \equiv 0$$

Eq. (6.23) may be written

$$a^F_j \leq a < a^R_j : q^{S,N}_{j,a} = \tilde{\beta}^{S,N}_{j,a,t(a)} + \beta^S_{a,t(a)} b^U_{j,a} \tag{6.26}$$

Pensions We derive the life-long spouse pension from eqs. (6.8) and (6.9) using the principles of precaution. For $a^F < a \leq a^L$ eq. (6.8) gives:

$$A^{B,S,N}_{j,a,a} = \left(1+i_{t(a)}\right)^{-1}\left(\tilde{b}^{S,N}_{\hat{j},a} + \sum_{y=a}^{a^L}\left(\frac{1}{1+g^S_{t(a)}}\right)^{y-a} b^{S,N}_{j,a,a}\frac{D_{y,t(a)}}{D_{a,t(a)}}\right) \Leftrightarrow$$

$$\left(1+i_{t(a)}\right) A^{B,S,N}_{j,y,y} - \tilde{b}^{S,N}_{j,a} = \sum_{y=a}^{a^L}\left(\frac{1}{1+g^S_{t(a)}}\right)^{y-a} b^{S,N}_{j,y,y}\frac{D_{y,t(a)}}{D_{a,t(a)}} \tag{6.27}$$

For $a^F < s < a$ we use eq. (6.9) to obtain:

$$A^{B,S,N}_{j,a,s} = \left(1+i_{t(a)}\right)^{-1}\left(\sum_{y=a}^{a^L}\left(\frac{1}{1+g^S_{t(a)}}\right)^{y-a} b^{S,N}_{j,a,s}\frac{D_{y,t(a)}}{D_{a,t(a)}}\right) \Leftrightarrow$$

$$\left(1+i_{t(a)}\right) A^{B,S,N}_{j,a,s} = \sum_{y=a}^{a^L}\left(\frac{1}{1+g^S_{t(a)}}\right)^{y-a} b^{S,N}_{j,a,s}\frac{D_{y,t(a)}}{D_{a,t(a)}} \tag{6.28}$$

Defining

$$a^F < a \leq a^L : \varphi^S_{a,t} \equiv \frac{1}{\sum_{y=a}^{a^L}\left(\frac{1}{1+g^S_t}\right)^{y-a}\frac{D_{y,t}}{D_{a,t}}} \tag{6.29}$$

eqs. (6.27) and (6.28) directly yield:

$$a^F < a \leq a^L : b^{S,N}_{j,a,a} = \varphi^S_{a,t(a)}\left(\left(1+i_{t(a)}\right) A^{B,S,N}_{j,a,a} - \tilde{b}^{S,N}_{j,a}\right) \tag{6.30}$$

$$a^F + 1 < a \leq a^L : b^{S,N}_{j,a,s} = \varphi^S_{a,t(a)}\left(1+i_{t(a)}\right) A^{B,S,N}_{j,a,s}, \quad a > s \tag{6.31}$$

In addition to this, first period spouse pensioners receive a one-time initial pension equal to $\tilde{b}^{S,N}_{j,a,t(a)}$.

Rather than using the assets at the beginning-of-period, $A_{j,a,s}^{B,S,N}$, we may also define the assets at the end of the period, $A_{j,a,s}^{S,N}$. We have

$$A_{j,a,a}^{S,N} = \left(1+i_{t(a)}\right) A_{j,a,a}^{B,S,N} - b_{j,a,a}^{S,A} - \tilde{b}_{j,a}^{S,N} \Leftrightarrow \tag{6.32}$$

$$\left(1+i_{t(a)}\right) A_{j,a,a}^{B,S,N} = A_{j,a,a}^{S,N} + b_{j,a,a}^{S,N} + \tilde{b}_{j,a}^{S,N} \tag{6.33}$$

and

$$a > s : A_{j,a,s}^{S,N} = \left(1+i_{t(a)}\right) A_{j,a,s}^{B,S,N} - b_{j,a,s}^{S,N} \Leftrightarrow \tag{6.34}$$

$$\left(1+i_{t(a)}\right) A_{j,a,s}^{B,S,N} = A_{j,a,s}^{S,N} + b_{j,a,s}^{S,N} \tag{6.35}$$

Inserting either eq. (6.33) into eq. (6.27) or eq. (6.35) into (6.28) yields

$$a^F < a < a^L : A_{j,a,s}^{S,N} = \sum_{y=a+1}^{a^L} \left(\frac{1}{1+g_{t(a)}^S}\right)^{y-a} b_{j,a,s}^{S,N} \frac{D_{y,t(a)}}{D_{a,t(a)}}, \quad a \geq s \tag{6.36}$$

Although end-of-period assets originate from past accumulation, eq. (6.28) shows that $A_{j,a,s}^{S,N}$ may be thought of as the present value (at age a) of the forecasted present value of *future* claims that the individual spouse pensioner has on the pension fund (or, equivalently, that spouse pensions are calculated such that the forecasted present value of future claims equal the assets). This interpretation is important when we are later to calculate the bonus of the pension fund.

Spouses of retired members

Premiums The spouse pension premium paid by a non-disabled retired member, i.e. a member of age $a \geq a_j^R$, is obtained in exactly the same way as was done for a non-retired member. We use the principles of precaution to form the forecast in eq. (6.10) to get

$$q_{j,a}^{S,R} = \frac{1}{1+\bar{\imath}_{t(a)}} \bar{r}_{a+1,t(a)}^M \bar{r}_{a+1,t(a)}^S \left(\tilde{b}_{j,a+1,t(a)}^{S,R} + \sum_{y=a+1}^{a^L} \left(\frac{1}{1+g_{t(a)}^S}\right)^{y-(a+1)} w_{y,t(a)}^S b_{j,a}^R \frac{D_{y,t(a)}}{D_{a+1,t(a)}} \right) \tag{6.37}$$

Compared with eq. (6.23) the only difference is that the one-time initial spouse pension is now $\tilde{b}_{j,a+1,t(a)}^{S,R}$ and the initital life-long spouse pension is given by $w_{y,t(a)}^S b_{j,a}^R$, which is based on the actual retirement pension of the deceased member instead of on the pension undertaking. Using $\beta_{a,t}^S$ defined in eq. (6.25) and the additional definition

$$a^{FR} \leq a < a^L : \tilde{\beta}_{j,a,t}^{S,R} \equiv \frac{1}{1+\bar{\imath}_t} \bar{r}_{a+1,t}^M \bar{r}_{a+1,t}^S \tilde{b}_{j,a+1,t}^{S,R} \tag{6.38}$$

$$y = a^L : \tilde{\beta}_{j,y,t}^{S,R} \equiv 0$$

we have

$$a_j^R \leq y \leq a^L : q_{j,a}^{S,R} = \tilde{\beta}_{j,a,t(a)}^{S,R} + \beta_{a,t(a)}^S b_{j,a}^R \tag{6.39}$$

note that $\beta_{a,t}^S = \tilde{\beta}_{j,y,t}^{S,R} \equiv 0$ for $a = a^L$ so no spouse pension premium is paid at age a^L.

Pensions In this case the spouse pension is derived in exactly the same way as was done in the case of a widow or widower who was married to a non-retired member. We apply the principles of precaution to form forecasts in eqs. (6.11) and (6.12). For $a_j^R < a \leq a^L$ we have:

$$A_{j,a,a}^{B,S,R} = \left(1 + i_{t(a)}\right)^{-1} \left(\tilde{b}_{j,a}^{S,R} + \sum_{y=a}^{a^L} \left(\frac{1}{1 + g_{t(a)}^S} \right)^{y-a} b_{j,a,a}^{S,R} \frac{D_{y,t(a)}}{D_{a,t(a)}} \right) \Leftrightarrow$$

$$\left(1 + i_{t(a)}^Z\right)^{-1} A_{j,a,a}^{B,S,R} = \tilde{b}_{j,a}^{S,D} + \sum_{y=a}^{a^L} \left(\frac{1}{1 + g_{t(a)}^S} \right)^{y-a} b_{j,a,a}^{S,R} \frac{D_{y,t(a)}}{D_{a,t(a)}} \tag{6.40}$$

While for $a_j^R < s < a$ eq. (6.12) implies

$$A_{j,a,s}^{B,S,R} = \left(1 + i_{t(a)}\right)^{-1} \sum_{y=a}^{a^L} \left(\frac{1}{1 + g_{t(a)}^S} \right)^{y-a} b_{j,a,s}^{S,R} \frac{D_{y,t(a)}}{D_{a,t(a)}} \Leftrightarrow$$

$$\left(1 + i_{t(a)}\right) A_{j,a,s}^{B,S,R} = \sum_{y=a}^{a^L} \left(\frac{1}{1 + g_{t(a)}^S} \right)^{y-a} b_{j,a,s}^{S,R} \frac{D_{y,t(a)}}{D_{a,t(a)}} \tag{6.41}$$

Using the definition of $\varphi_{a,t}^S$ in eq. (6.29), eqs. (6.40) and (6.41) now yield:

$$a_j^R < a \leq a^L : b_{j,a,a}^{S,R} = \varphi_{a,t(a)}^S \left(\left(1 + i_{t(a)}\right) A_{j,a,a}^{B,S,R} - \tilde{b}_{j,a}^{S,R} \right) \tag{6.42}$$

$$a_j^R + 1 < a \leq a^L : f_{j,a,s}^{S,R} = \varphi_{a,t(a)}^S \left(1 + i_{t(a)}\right) A_{j,a,s}^{B,S,R}, \; a > s \tag{6.43}$$

In addition to this, first period spouse pensioners receive a one-time initial pension equal to $\tilde{b}_{j,a,t(a)}^{S,R}$.

Defining end-of-period assets, $A_{j,a,s}^{S,R}$, as

$$A_{j,a,a}^{S,R} = \left(1 + i_{t(a)}\right) A_{j,a,a}^{B,S,R} - b_{j,a,a}^{S,R} - \tilde{b}_{j,a,t(a)}^{S,R} \Leftrightarrow \tag{6.44}$$

$$\left(1 + i_{t(a)}\right) A_{j,a,a,r}^{B,S,R} = A_{j,a,a}^{S,R} + b_{j,a,a}^{S,R} + \tilde{b}_{j,a,t(a)}^{S,R} \tag{6.45}$$

and

$$a > s : A_{j,a,s}^{S,R} = \left(1 + i_{t(a)}\right) A_{j,a,s}^{B,S,R} - b_{j,a,s}^{S,R} \Leftrightarrow \tag{6.46}$$

$$\left(1 + i_{t(a)}\right) A_{j,a,s}^{B,S,R} = A_{j,a,s}^{S,R} + b_{j,a,s}^{S,R} \tag{6.47}$$

Insertion of eq. (6.45) into eq. (6.40) or eq. (6.47) into eq. (6.41) yields

$$a^F + 1 < a < a^L : A^{S,R}_{j,a,s} = \sum_{y=a+1}^{A^L} \left(\frac{1}{1+g^S_{t(a)}} \right)^{y-a} b^{S,R}_{j,a,s} \frac{D_{y,t(a)}}{D_{a,t(a)}}, \quad a \geq s \tag{6.48}$$

Just as in the case of spouse pensioners who were married to non-retired members, eq. (6.48) shows that $A^{ZF,S,R}_{j,y,s}$ may be given the interpretation of the forecasted present value (at age a) of the *future* claim that the individual spouse pensioner has on the pension fund.

Spouses of disabled members

Premiums By using the principles of precaution to form the forecast in eq. (6.13) we obtain:

$$q^{S,D}_{j,a,d} = \frac{1}{1+\bar{\imath}_{t(a)}} \bar{r}^M_{a+1,t(a)} \bar{r}^S_{a+1,t(a)} \left[\tilde{b}^{S,D}_{j,a+1,t(a)} + \sum_{y=a+1}^{a^L} \left(\frac{1}{1+g^S_{t(a)}} \right)^{y-(a+1)} w^S_{y,t(a)} b^D_{j,a,d} \frac{D_{y,t(a)}}{D_{a+1,t(a)}} \right] \tag{6.49}$$

In this case the one-time inital spouse pension is given by $\tilde{b}^{S,D}_{j,a+1,t(a)}$, while the life-long annual pension is initialy given by $w^S_{y,t(a)} b^D_{j,a,d}$, which constitute the only differences compared with the other types of spouse pensions premiums.

Using $\beta^S_{a,t}$ defined in eq. (6.25) and the additional definition

$$\begin{aligned} a^F \leq a < a^L &: \tilde{\beta}^{S,D}_{j,a,t} \equiv \frac{1}{1+\bar{\imath}_t} \bar{r}^M_{a+1,t} \bar{r}^S_{a+1,t} \tilde{b}^{S,D}_{j,a+1,t} \\ y = a^L &: \tilde{\beta}^{S,D}_{j,y,t} \equiv 0 \end{aligned} \tag{6.50}$$

we have

$$a^F_j < a \leq a^L : q^{S,D}_{j,a,d} = \tilde{\beta}^{S,D}_{j,a,t(a)} + \beta^S_{a,t(a)} b^D_{j,a,d} \tag{6.51}$$

Note that $\beta^S_{a,t} \equiv 0$ for $a = a^L$ so no spouse pension premium is paid at age a^L.

Pensions In this case we apply the principles of precaution to form forecasts in eqs. (6.14) and (6.15). For $a^F + 1 < a \leq a^L$ we have:

$$A^{B,S,D}_{j,a,a,d} = \left(1+i_{t(a)}\right)^{-1} \left(\tilde{b}^{S,D}_{j,a} + \sum_{y=a}^{a^L} \left(\frac{1}{1+g^S_{t(a)}} \right)^{y-a} b^{S,D}_{j,a,a,d} \frac{D_{y,t(a)}}{D_{a,t(a)}} \right) \Leftrightarrow$$

$$\left(1+i^Z_{t(a)}\right)^{-1} A^{B,S,D}_{j,a,a,d} = \tilde{b}^{S,D}_{j,a} + \sum_{y=a}^{a^L} \left(\frac{1}{1+g^S_{t(a)}} \right)^{y-a} b^{S,D}_{j,a,a,d} \frac{D_{y,t(a)}}{D_{a,t(a)}} \tag{6.52}$$

While for $a^F + 1 < s < a$ eq. (6.15) implies

$$A^{B,S,D}_{j,a,s,d} = \left(1 + i_{t(a)}\right)^{-1} \sum_{y=a}^{a^L} \left(\frac{1}{1 + g^S_{t(a)}}\right)^{y-a} b^{S,D}_{j,a,s,d} \frac{D_{y,t(a)}}{D_{a,t(a)}} \Leftrightarrow$$

$$\left(1 + i_{t(a)}\right) A^{B,S,D}_{j,a,s,d} = \sum_{y=a}^{a^L} \left(\frac{1}{1 + g^S_{t(a)}}\right)^{y-a} b^{S,D}_{j,a,s,d} \frac{D_{y,t(a)}}{D_{a,t(a)}} \tag{6.53}$$

Using the definition of $\varphi^S_{a,t}$ in eq. (6.29) eqs. (6.52) and (6.53) now yield:

$$a^F + 1 < a \leq a^L : b^{S,D}_{j,a,a,d} = \varphi^S_{a,t(a)} \left(\left(1 + i_{t(a)}\right) A^{B,S,D}_{j,a,a,d} - \tilde{b}^{S,D}_{j,a}\right) \tag{6.54}$$

$$a^F + 2 < a \leq a^L : f^{S,D}_{j,a,s,d} = \varphi^S_{a,t(a)} \left(1 + i_{t(a)}\right) A^{B,S,D}_{j,a,s,d},\ a > s \tag{6.55}$$

In addition to this, first period spouse pensioners receive a one-time initial pension equal to $\tilde{b}^{S,D}_{j,a,t(a)}$.

Defining end-of-period assets, $A^{S,D}_{j,a,s,d}$, as

$$A^{S,D}_{j,a,a,d} = \left(1 + i_{t(a)}\right) A^{B,S,D}_{j,a,a,d} - b^{S,D}_{j,a,a,d} - \tilde{b}^{S,D}_{j,a,t(a)} \Leftrightarrow \tag{6.56}$$

$$\left(1 + i_{t(a)}\right) A^{B,S,D}_{j,a,a,d} = A^{S,D}_{j,a,a,d} + b^{S,D}_{j,a,a,d} + \tilde{b}^{S,D}_{j,a,t(a)} \tag{6.57}$$

and

$$a > s : A^{S,D}_{j,a,s,d} = \left(1 + i_{t(a)}\right) A^{B,S,D}_{j,a,s,d} - b^{S,D}_{j,a,s,d} \Leftrightarrow \tag{6.58}$$

$$\left(1 + i_{t(a)}\right) A^{B,S,D}_{j,a,s,d} = A^{S,D}_{j,a,s,d} + b^{S,D}_{j,a,s,d} \tag{6.59}$$

Insertion of eq. (6.57) into eq. (6.52) or eq. (6.59) into eq. (6.53) yields

$$a^F + 1 < a < a^L : A^{S,D}_{j,a,s,d} = \sum_{y=a+1}^{A^L} \left(\frac{1}{1 + g^S_{t(a)}}\right)^{y-a} b^{S,D}_{j,a,s,d} \frac{D_{y,t(a)}}{D_{a,t(a)}}, \quad a \geq s \tag{6.60}$$

$A^{ZF,S,D}_{j,y,s,d}$ may thus be given the interpretation of the present value (at age a) of the forecasted *future* claims that the individual spouse pensioner has on the pension fund.

6.3.2 Disablement pensions

Premiums Applying the rules of precaution to equation (6.16) yields the disablement pension premium:

$$q^D_{j,a} = \frac{1}{1 + \bar{\imath}_{t(a)}} \bar{r}^D_{a+1,t(a)} \left(1 - \bar{r}^M_{a+1,t(a)}\right) \tag{6.61}$$

$$\times \left[\tilde{b}^D_{j,a+1,t(a)} + \sum_{y=a+1}^{a^L} \tilde{\beta}^{S,D}_{j,y,t(a)} \frac{D_{y,t(a)}}{D_{a+1,t(a)}} + \sum_{y=a+1}^{a^L} \left(1 + \beta^S_{y,t(a)}\right) w^D_{y,t(a)} b^U_{j,a} \frac{D_{y,t(a)}}{D_{a+1,t(a)}}\right]$$

Where we have also used eq. (6.51), which gives the spouse pension premium for disabled members. The last two terms on the right hand side of the first line give the probability that member j will survive but become disabled before turning $a+1$ years old. The second line is the forecasted total discounted cost associated with the disablement pension, both including the one-time initial pension, the annual disablement pension and the premiums to spouse pension insurance. This cost is measured in present value at age $a+1$ and therefore the term including the base interest rate in the first line is used to discount the cost to age a where the premium is paid.

We now define

$$\begin{aligned} a^F &\leq a < a^L : \\ \tilde{\beta}^D_{j,a,t} &\equiv \frac{1}{1+\bar{\imath}_t}\bar{r}^D_{a+1,t}\left(1-\bar{r}^M_{a+1,t}\right)\left(\tilde{b}^D_{j,a+1,t} + \sum_{y=a+1}^{a^L}\tilde{\beta}^{S,D}_{j,y,t}\frac{D_{y,t}}{D_{a+1,t}}\right) \qquad (6.62) \\ a^F &\leq y < a^L : \\ \beta^D_{a,t} &\equiv \frac{1}{1+\bar{\imath}_t}\bar{r}^D_{a+1,t}\left(1-\bar{r}^M_{a+1,t}\right)\sum_{y=a+1}^{a^L}\left(1+\beta^S_{y,t}\right)w^D_{y,t}\frac{D_{y,t}}{D_{a+1,t}} \qquad (6.63) \end{aligned}$$

This enables us to write the disablement premium of eq. (6.61) as

$$a^F_j \leq a < \min\left[a^R_j, a^R\right] : q^D_{j,a} = \tilde{\beta}^D_{j,a,t(a)} + \beta^D_{a,t(a)}b^U_{j,a} \qquad (6.64)$$

Pensions The principles of precautions applied to the budget constraint for a disablement pensioner, eq. (6.17), now yield:

$$\begin{aligned} A^{B,D}_{j,a,a} &= \left(1+i_{t(a)}\right)^{-1}\left(\tilde{b}^D_{j,a,t(a)} + \sum_{y=a}^{a^L}\left(\tilde{\beta}^{S,D}_{j,y,t(a)} + \left(1+\beta^S_{y,t(a)}\right)b^D_{j,a,a}\right)\frac{D_{y,t(a)}}{D_{a,t(a)}}\right) \Leftrightarrow \\ \left(1+i_{t(a)}\right)A^{B,D}_{j,a,a} &= \tilde{b}^D_{j,a,t(a)} + \sum_{y=a}^{a^L}\tilde{\beta}^{S,D}_{j,y,t(a)}\frac{D_{y,t(a)}}{D_{a,t(a)}} + \sum_{y=a}^{a^S}\left(1+\beta^S_{y,t(a)}\right)b^D_{j,a,a}\frac{D_{y,t(a)}}{D_{a,t(a)}} \qquad (6.65) \end{aligned}$$

where also the spouse pension premium of eq. (6.51) has been used.

For $d < a$ we obtain from the budget constraint in eq. (6.18) and the spouse pension premium in (6.51):

$$\begin{aligned} A^{B,D}_{j,a,d} &= \left(1+i_{t(a)}\right)^{-1}\sum_{y=a}^{a^L}\left(\tilde{\beta}^{S,D}_{j,y,t(a)} + \left(1+\beta^S_{y,t(a)}\right)b^D_{j,a,d}\right)\frac{D_{y,t(a)}}{D_{a,t(a)}} \Leftrightarrow \\ \left(1+i_{t(a)}\right)A^{B,D}_{j,a,d} &= \sum_{y=a}^{a^L}\tilde{\beta}^{S,D}_{j,y,t(a)}\frac{D_{y,t(a)}}{D_{a,t(a)}} + \sum_{y=a}^{a^L}\left(1+\beta^S_{y,t(a)}\right)b^D_{j,a,d}\frac{D_{y,t(a)}}{D_{a,t(a)}} \qquad (6.66) \end{aligned}$$

Defining

$$a^F < a \leq a^L : \tilde{\varphi}^D_{j,a,t} \equiv \frac{\sum_{y=a}^{a^L} \tilde{\beta}^{S,D}_{j,y,t} \frac{D_{y,t}}{D_{a,t}}}{\sum_{y=a}^{a^L} \left(1+\beta^S_{y,t}\right) \frac{D_{y,t}}{D_{a,t}}} \tag{6.67}$$

$$a^F < a \leq a^L : \varphi^D_{a,t} \equiv \frac{1}{\sum_{y=a}^{a^L} \left(1+\beta^S_{y,t}\right) \frac{D_{y,t}}{D_{a,t}}} \tag{6.68}$$

eqs. (6.65) and (6.66) may be written as:

NB: ÆNDRET I.F.T. DREAM

$$b^D_{j,a,a} = \varphi^D_{a,t(a)} \left(\left(1+i_{t(a)}\right) A^{B,D}_{j,a,a} - \tilde{b}^D_{j,a,t(a)} \right) - \tilde{\varphi}^D_{j,a,t(a)} \tag{6.69}$$

$$b^D_{j,a,d} = \varphi^D_{a,t(a)} \left(1+i_{t(a)}\right) A^{B,D}_{j,a,d} - \tilde{\varphi}^D_{j,a,t(a)}, \quad a > d \tag{6.70}$$

In addition to this, first period disablement pensioners receive a one-time initial pension equal to $\tilde{b}^D_{j,a,t(a)}$.

For later use we define end of period assets $A^D_{j,a,d}$ in terms of beginning of period assets, $A^{B,D}_{j,a,d}$. We have

$$A^D_{j,a,a} = \left(1+i_{t(a)}\right) A^{B,D}_{j,a,a} - b^D_{j,a,a} - \tilde{b}^D_{j,a,t(a)} - q^{S,D}_{j,a,a} \Leftrightarrow \tag{6.71}$$

$$\left(1+i_{t(a)}\right) A^{B,D}_{j,a,a} = A^D_{j,a,a} + \left(1+\beta^S_{a,t(a)}\right) b^D_{j,a,a} + \tilde{b}^D_{j,a,t(a)} + \tilde{\beta}^{S,D}_{j,a,t(a)} \tag{6.72}$$

and

$$a > d : A^D_{j,a,d} = \left(1+i_{t(a)}\right) A^{B,D}_{j,a,d} - b^D_{j,a,d} - q^{S,D}_{j,a,d} \Leftrightarrow \tag{6.73}$$

$$\left(1+i_{t(a)}\right) A^{B,D}_{j,a,d} = A^D_{j,a,d} + \left(1+\beta^S_{a,t(a)}\right) b^D_{j,a,d} + \tilde{\beta}^{S,D}_{j,a,t(a)} \tag{6.74}$$

Inserting eq. (6.72) into eq. (6.65) or (6.74) into eq. (6.66) we obtain

$$a^F < a < a^L : A^D_{j,a,d} = \sum_{y=a+1}^{a^L} \tilde{\beta}^{S,D}_{j,y,t(a)} \frac{D_{y,t(a)}}{D_{a,t(a)}} + \sum_{y=a+1}^{A^L} \left(1+\beta^S_{y,t(a)}\right) b^D_{j,a,d} \frac{D_{y,t(a)}}{D_{a,t(a)}}, \quad a \geq d \tag{6.75}$$

Once more it is seen that end-of-period assets are equal to the present value (at age a) of the forecasted *future* claims that the individual disablement pensioner has on the pension fund.

6.3.3 Pension undertakings

Pension undertakings are defined as the forecasted value of the retirement pension provided that the member retires at the forecasted retirement age and that no one-time initial retirement pension is chosen. Therefore pension undertakings are calculated from the budget constraints of non-retired active members using the principles of precautions and equations (6.26) and (6.64), giving spouse and disablement pension premiums. For $a_j^F \leq a < \min\left[a_j^R, a^R\right]$ we get use the budget constraint in eq. (6.1) to get:

$$
\begin{aligned}
& A_{j,a}^{B,N} + \tilde{A}_{j,a}^{B,N} \\
& + \left(1+i_{t(a)}\right)^{-1} \left[\sum_{y=a}^{a^R-1} \left(q_{j,a} - \left(\tilde{\beta}_{j,y,t(a)}^{D} + \beta_{y,t(a)}^{D} b_{j,a}^{U} \right) - \left(\tilde{\beta}_{j,y,t(a)}^{S,N} + \beta_{y,t(a)}^{S} b_{j,a}^{U} \right) \right) \frac{D_{y,t(a)}^{A}}{D_{a,t(a)}^{A}} \right] \\
= \; & \left(1+i_{t(a)}\right)^{-1} \left[\sum_{a=a^R}^{a^L} \left(b_{j,a}^{U} + \tilde{\beta}_{j,y,t(a)}^{S,R} + \beta_{y,t(a)}^{S} b_{j,a}^{U} \right) \frac{D_{y,t(a)}^{A}}{D_{a,t(a)}^{A}} \right] \Leftrightarrow
\end{aligned}
$$

$$
\begin{aligned}
& \left(1+i_{t(a)}\right) \left(A_{j,a}^{B,N} + \tilde{A}_{j,a}^{B,N} \right) + \sum_{y=a}^{a^R-1} q_{j,a} \frac{D_{y,t(a)}^{A}}{D_{a,t(a)}^{A}} \\
& - \sum_{y=a}^{a^R-1} \left(\tilde{\beta}_{j,y,t(a)}^{D} + \tilde{\beta}_{j,y,t(a)}^{S,N} \right) \frac{D_{y,t(a)}^{A}}{D_{a,t(a)}^{A}} - \sum_{y=a}^{a^R-1} \left(\beta_{y,t(a)}^{D} + \beta_{y,t(a)}^{S} \right) b_{j,a}^{U} \frac{D_{y,t(a)}^{A}}{D_{a,t(a)}^{A}} \\
= \; & \sum_{y=a^R}^{a^L} \tilde{\beta}_{j,y,t(a)}^{S,R} \frac{D_{y,t(a)}^{A}}{D_{a,t(a)}^{A}} + \sum_{y=a^R}^{a^L} \left(1+\beta_{y,t(a)}^{S}\right) b_{j,a}^{U} \frac{D_{y,t(a)}^{A}}{D_{a,t(a)}^{A}}
\end{aligned} \tag{6.76}
$$

For $a^R \leq y < a_j^R$ we use the budget constraint in eq. (6.2):

$$
\begin{aligned}
& A_{j,a}^{B,N} + \tilde{A}_{j,a}^{B,N} + \left(1+i_{t(a)}\right)^{-1} \left(q_{j,a} - \left(\tilde{\beta}_{j,a,t(a)}^{S,N} + \beta_{at(a)}^{S} b_{j,a}^{U} \right) \right) \\
= \; & \left(1+i_{t(a)}\right)^{-1} \sum_{y=a+1}^{a^L} \left(b_{j,a}^{U} + \tilde{\beta}_{j,y,t(a)}^{S,R} + \beta_{y,t(a)}^{S} b_{j,a}^{U} \right) \frac{D_{y,t(a)}}{D_{a,t(a)}} \Leftrightarrow
\end{aligned}
$$

$$
\begin{aligned}
& \left(1+i_{t(a)}\right) \left(A_{j,a}^{B,N} + \tilde{A}_{j,a}^{B,N} \right) + q_{j,a} - \left(\tilde{\beta}_{j,a,t(a)}^{S,N} + \beta_{a,t(a)}^{S} b_{j,a}^{U} \right) \\
= \; & \sum_{y=a+1}^{a^L} \tilde{\beta}_{j,y,t(a)}^{S,R} \frac{D_{y,t(a)}}{D_{a,t(a)}} + \sum_{y=a+1}^{a^L} \left(1+\beta_{y,t(a)}^{S}\right) b_{j,a}^{U} \frac{D_{y,t(a)}}{D_{a,t(a)}}
\end{aligned} \tag{6.77}
$$

Defining

$$a^F \leq a < a^R : \varphi^{U,A}_{a,t} \equiv \frac{1}{\sum_{y=a}^{a^R-1}\left(\beta^D_{y,t}+\beta^S_{y,t}\right)\frac{D^A_{y,t}}{D^A_{a,t}}+\sum_{y=a^R}^{a^L}\left(1+\beta^S_{y,t}\right)\frac{D^A_{y,t}}{D^A_{a,t}}} \tag{6.78}$$

$$a^R \leq a < a^{LR} : \varphi^{U,A}_{a,t} \equiv \frac{1}{\beta^S_{a,t}+\sum_{y=a+1}^{a^L}\left(1+\beta^S_{y,t}\right)\frac{D_{y,t}}{D_{a,t}}} \tag{6.79}$$

$$a^F \leq a < a^R : \varphi^{U,q}_{a,t} \equiv \frac{\sum_{y=a}^{a^R-1}\frac{D^A_{y,t}}{D^A_{a,t}}}{\sum_{y=a}^{a^R-1}\left(\beta^D_{y,t}+\beta^S_{y,t}\right)\frac{D^A_{y,t}}{D^A_{a,t}}+\sum_{y=a^R}^{a^L}\left(1+\beta^S_{y,t}\right)\frac{D^A_{y,t}}{D^A_{a,t}}} \tag{6.80}$$

$$a^R \leq a < a^{LR} : \varphi^{U,q}_{a,t} \equiv \frac{1}{\beta^S_{a,t}+\sum_{y=a+1}^{a^L}\left(1+\beta^S_{y,t}\right)\frac{D_{y,t}}{D_{a,t}}} \tag{6.81}$$

$$a^F \leq a < a^R : \tilde{\varphi}^U_{j,a,t} \equiv \frac{\sum_{y=a}^{a^R-1}\left(\tilde{\beta}^D_{j,y,t}+\tilde{\beta}^{S,A}_{j,y,t}\right)\frac{D^A_{y,t}}{D^A_{a,t}}+\sum_{y=a^R}^{a^{Last}}\tilde{\beta}^{S,R}_{j,y,t}\frac{D^A_{y,t}}{D^A_{a,t}}}{\sum_{y=a}^{a^R-1}\left(\beta^D_{y,t}+\beta^S_{y,t}\right)\frac{D^A_{y,t}}{D^A_{a,t}}+\sum_{y=a^R}^{a^L}\left(1+\beta^S_{y,t}\right)\frac{D^A_{y,t}}{D^A_{a,t}}} \tag{6.82}$$

$$a^R \leq a < a^{LR} : \tilde{\varphi}^U_{j,a,t} \equiv \frac{\sum_{y=a}^{a^L}\tilde{\beta}^{S,A}_{j,y,t}\frac{D_{y,t}}{D_{a,t}}}{\beta^S_{a,t}+\sum_{y=a+1}^{a^L}\left(1+\beta^S_{y,t}\right)\frac{D_{y,t}}{D_{a,t}}} \tag{6.83}$$

both equtions (6.76) and (6.77) may be written as:

$$\begin{aligned} a^F_j &\leq y < a^R_j : \\ b^U_{j,a} &= \varphi^{U,A}_{a,t(a)}\left(1+i_{t(a)}\right)\left(A^{B,N}_{j,a}+\tilde{A}^{B,N}_{j,a}\right)+\varphi^{U,q}_{a,t(a)}q_{j,a}-\tilde{\varphi}^U_{j,a,t(a)} \end{aligned} \tag{6.84}$$

In addition to, this we also want to calculate a forecasted once–time initial retirement pension , which may be paid at the retirement age. For $a < \min\left[a^R_j, a^R\right]$ using eq. (6.3) we obtain (using the principles of precaution):

$$\begin{aligned} \tilde{A}^{B,N}_{j,a}+\left(1+i_{t(a)}\right)^{-1}\sum_{y=a}^{a^R-1}\tilde{q}_{j,a}\frac{D^A_{y,t(a)}}{D^A_{a,t(a)}} &= \left(1+i_{t(a)}\right)^{-1}\tilde{b}^U_{j,a}\frac{D^A_{a^R,t(a)}}{D^A_{a,t(a)}} \Leftrightarrow \\ \left(1+i_{t(a)}\right)\tilde{A}^{B,N}_{j,a}+\sum_{y=a}^{a^R-1}\tilde{q}_{j,a}\frac{D^A_{y,t(a)}}{D^A_{a,t(a)}} &= \tilde{b}^U_{j,a}\frac{D^A_{a^R,t(a)}}{D^A_{a,t(a)}} \end{aligned} \tag{6.85}$$

For $a^R \leq a < a^R_j$ we obtain

$$\begin{aligned} \tilde{A}^{B,N}_{j,a}+\tilde{q}_{j,a}\left(1+i_{t(a)}\right)^{-1} &= \left(1+i_{t(a)}\right)^{-1}\tilde{b}^U_{j,a}\frac{D_{a+1,t(a)}}{D_{a,t(a)}} \Leftrightarrow \\ \left(1+i_{t(a)}\right)\tilde{A}^{B,N}_{j,a}+\tilde{q}_{j,a} &= \tilde{b}^U_{j,a}\frac{D_{a+1,t(a)}}{D_{a,t(a)}} \end{aligned} \tag{6.86}$$

Defining

$$a^F \leq a < a^R: \qquad \tilde{\varphi}_{a,t}^{U,A} \equiv \frac{1}{\frac{D_{a^R,t}^A}{D_{a,t}^A}} \tag{6.87}$$

$$a^R \leq a < a^{LR}: \qquad \tilde{\varphi}_{a,t}^{U,A} \equiv \frac{1}{\frac{D_{a+1,t}}{D_{a,t}}} \tag{6.88}$$

$$a^F \leq a < a^R: \qquad \tilde{\varphi}_{a,t}^{U,q} \equiv \frac{\sum_{y=a}^{a^R-1} \frac{D_{y,t}^A}{D_{a,t}^A}}{\frac{D_{a^R,t}^A}{D_{a,t}^A}} \tag{6.89}$$

$$a^R \leq a < a^{LR}: \qquad \tilde{\varphi}_{a,t}^{U,q} \equiv \frac{1}{\frac{D_{a+1,t}}{D_{a,t}}} \tag{6.90}$$

Both equations (6.85) and (6.86) may be written as:

$$a_j^F \leq a < a_j^R : \tilde{b}_{j,a}^U = \tilde{\varphi}_{a,t(a)}^{U,A} \left(1 + i_{t(a)}\right) \tilde{A}_{j,a}^{B,N} + \tilde{\varphi}_{a,t(a)}^{U,q} \tilde{q}_{j,a} \tag{6.91}$$

Now turning to the definition of end-of-period assets we begin with assets for one-time initial retirement pensions: For $a < a_j^R$ we define

$$\tilde{A}_{j,a}^N = \left(1 + i_{t(a)}\right) \tilde{A}_{j,a}^{B,N} + \tilde{q}_{j,a} \Leftrightarrow \tag{6.92}$$

$$\left(1 + i_{t(a)}\right) \tilde{A}_{j,a}^{B,N} = \tilde{A}_{j,a}^N - \tilde{q}_{j,a} \tag{6.93}$$

Inserting eq. (6.93) in eq. (6.85) for $a < a^R - 1$ yields

$$a < a^R - 1 : \tilde{A}_{j,a}^N = \tilde{b}_{j,a}^U \frac{D_{a^R,t(a)}^A}{D_{a,t(a)}^A} - \sum_{y=a+1}^{a^R-1} \tilde{q}_{j,a} \frac{D_{y,t(a)}^A}{D_{a,t(a)}^A} \tag{6.94}$$

Inserting eq. (6.93) in eq. (6.85) for $a = a^R - 1$ yields

$$y = a^R - 1 : \tilde{A}_{j,a}^N = \tilde{b}_{j,a}^U \frac{D_{a^R,t(a)}^A}{D_{a,t(a)}^A} \tag{6.95}$$

Finally, inserting eq. (6.93) in eq. (6.86) for $a^R \leq a < a_j^R$ yields:

$$a^R \leq a < a_j^R : \tilde{A}_{j,a}^N = \tilde{b}_{j,a}^U \frac{D_{a+1,t(a)}}{D_{a,t(a)}} \tag{6.96}$$

In all three cases we thus obtain that end-of-period-assets are equal to the present value of expected future claims.

We now define total end-of-period assets for non-retired active members as, $A_{j,y}^N + \tilde{A}_{j,y}^N$ for $a < \min\left[a_j^R, a^R\right]$ as

$$A_{j,a}^N + \tilde{A}_{j,a}^N = \left(1 + i_{t(a)}\right) \left(A_{j,a}^{B,N} + \tilde{A}_{j,a}^{B,N}\right) + q_{j,a} - q_{j,a}^D - q_{j,a}^{S,N} \Leftrightarrow \tag{6.97}$$

$$\begin{aligned}
&\left(1+i_{t(a)}\right)\left(A^{B,N}_{j,a}+\tilde{A}^{B,N}_{j,a}\right)\\
=\;& A^{N}_{j,a}+\tilde{A}^{N}_{j,y}\\
&-\left(q_{j,a}-\left(\tilde{\beta}^{D}_{j,a,t(a)}+\beta^{D}_{a,t(a)}b^{U}_{j,a}\right)-\left(\tilde{\beta}^{S,N}_{j,a,t(a)}+\beta^{S}_{a,t(a)}b^{U}_{j,a}\right)\right)
\end{aligned} \tag{6.98}$$

Inserting eq. (6.98) in eq. (6.76) we obtain for $a < a^R - 1 \wedge a < a^R_j$:

$$\begin{aligned}
a \;&<\; a^R - 1 \wedge a < a^R_j :\\
&A^{N}_{j,a}+\tilde{A}^{N}_{j,a}\\
=\;& \sum_{y=a+1}^{a^R-1}\left(\tilde{\beta}^{D}_{j,y,t(a)}+\tilde{\beta}^{S,N}_{j,y,t(a)}\right)\frac{D^{A}_{y,t(a)}}{D^{A}_{a,t(a)}}+\sum_{y=a+1}^{a^R-1}\left(\beta^{D}_{y,t(a)}+\beta^{S}_{y,t(a)}\right)b^{U}_{j,a}\frac{D^{A}_{y,t(a)}}{D^{A}_{a,t(a)}}\\
&+\sum_{y=a^R}^{a^L}\tilde{\beta}^{S,R}_{j,y,t(a)}\frac{D^{A}_{y,t(a)}}{D^{A}_{a,t(a)}}+\sum_{y=a^R}^{a^L}\left(1+\beta^{S}_{y,t(a)}\right)b^{U}_{j,a}\frac{D^{A}_{y,t(a)}}{D^{A}_{a,t(a)}}\\
&-\sum_{y=a+1}^{a^R-1}q_{j,a}\frac{D^{A}_{y,t(a)}}{D^{A}_{a,t(a)}}
\end{aligned} \tag{6.99}$$

Inserting eq. (6.97) in eq. (6.76) for $a^R - 1 = a < a^R_j$ we obtain:

$$\begin{aligned}
a^R - 1 \;&=\; a < a^R_j :\\
&A^{N}_{j,a}+\tilde{A}^{N}_{j,a}\\
=\;& \sum_{y=a^R}^{a^L}\tilde{\beta}^{S,R}_{j,y,t(a)}\frac{D^{A}_{y,t(a)}}{D^{A}_{a,t(a)}}+\sum_{y=a^R}^{a^L}\left(1+\beta^{S}_{y,t(a)}\right)b^{U}_{j,a}\frac{D^{A}_{y,t(a)}}{D^{A}_{a,t(a)}}
\end{aligned} \tag{6.100}$$

For $a^R \leq a < a^R_j$ we define total end of period assets, $A^N_{j,a} + \tilde{A}^N_{j,a}$ as

$$A^{N}_{j,a}+\tilde{A}^{N}_{j,a}=\left(1+i_{t(a)}\right)\left(A^{B,N}_{j,a}+\tilde{A}^{B,N}_{j,a}\right)+q_{j,a}-q^{S,N}_{j,a}\Leftrightarrow \tag{6.101}$$

$$\begin{aligned}
&\left(1+i_{t(a)}\right)\left(A^{B,N}_{j,a}+\tilde{A}^{B,N}_{j,a}\right)\\
=\;& A^{N}_{j,a}+\tilde{A}^{N}_{j,a}-\left(q_{j,a}-\left(\tilde{\beta}^{S,N}_{j,a,t(a)}+\beta^{S}_{a,t(a)}b^{U}_{j,a}\right)\right)
\end{aligned} \tag{6.102}$$

Inserting eq. (6.102) in eq. (6.77) we obtain:

$$\begin{aligned}
a^R \;&\leq\; a < a^R_j\\
A^{N}_{j,a}+\tilde{A}^{N}_{j,a} \;&=\; \sum_{y=a+1}^{a^L}\tilde{\beta}^{S,R}_{j,y,t(a)}\frac{D_{y,t(a)}}{D_{a,t(a)}}+\sum_{y=a+1}^{a^L}\left(1+\beta^{S}_{y,t(a)}\right)b^{U}_{j,a}\frac{D_{y,t(a)}}{D_{a,t(a)}}
\end{aligned} \tag{6.103}$$

Eqs. (6.99), (6.100) and (6.103) thus all state the same; total end-of-period assets may be thought of as the discounted value of all forecasted *future* claims that the non-retired member has on the pension fund.

6.3.4 Retirement pensions

We now finally consider a retired member. If the member has just retired, i.e. if $a = a_j^R$, the member may choose to receive a part of the retirement pension as one-time pension. From eq. (6.4) we obtain

$$a = a_j^R : \tilde{b}_{j,a}^R = \left(1 + i_{t(a)}\right) \tilde{d}_{j,a}^R \tilde{A}_{j,a}^{B,R} \tag{6.104}$$

where $\tilde{d}_j^R = 1$, if the member chooses the pension, and $\tilde{d}_j^R = 0$ otherwise.

In addition to this, a lifelong annual pension, $b_{j,a}^R$ is initiated and from eq. (6.5) we obtain (using the principles of precaution and eq. (6.39) giving spouse pension premiums for retired, active members)

$$\begin{aligned} & A_{j,a}^{B,R} + \left(1 - \tilde{d}_{j,a}^R\right) \tilde{A}_{j,a}^{B,R} \\ = \; & \left(1 + i_{t(a)}\right)^{-1} \sum_{y=a}^{a^L} \left(b_{j,a}^R + \tilde{\beta}_{j,y,t(a)}^{S,R} + \beta_{y,t(a)}^S b_{j,t(a)}^R\right) \frac{D_{y,t(a)}}{D_{a,t(a)}} \Leftrightarrow \end{aligned}$$

$$\begin{aligned} & \left(1 + i_{t(a)}\right) \left(A_{j,a}^{B,R} + \left(1 - \tilde{d}_{j,a}^R\right) \tilde{A}_{j,a}^{B,R}\right) \\ = \; & \sum_{y=a}^{a^L} \tilde{\beta}_{j,y,t(a)}^{S,R} \frac{D_{y,t(a)}}{D_{a,t(a)}} + \sum_{y=a}^{a^L} \left(1 + \beta_{y,t(a)}^S\right) b_{j,a}^R \frac{D_{y,t(a)}}{D_{a,t(a)}} \end{aligned} \tag{6.105}$$

Defining

$$\varphi_{a,t}^{R,A} \equiv \frac{1}{\sum_{y=a}^{a^L} \left(1 + \beta_{y,t}^S\right) \frac{D_{y,t}}{D_{a,t}}} \tag{6.106}$$

$$\tilde{\varphi}_{j,a,t}^R \equiv \frac{\sum_{y=a}^{a^L} \tilde{\beta}_{j,y,t}^{S,R} \frac{D_{y,t}}{D_{a,t}}}{\sum_{y=a}^{a^L} \left(1 + \beta_{y,t}^S\right) \frac{D_{y,t}}{D_{a,t}}} \tag{6.107}$$

we may write the initial retirement period life-long annual retirement pension as

$$a = a_j^R : b_{j,a}^R = \varphi_{a,t(a)}^{R,A} \left(1 + i_{t(a)}\right) \left(A_{j,a}^{B,R} + \left(1 - \tilde{d}_{j,a}^R\right) \tilde{A}_{j,a}^{B,R}\right) - \tilde{\varphi}_{j,a,t(a)}^R \tag{6.108}$$

Finally, for $a_j^R < a \leq a^L$ we obtain from eq. (6.6) and:

$$\begin{aligned} A_{j,a}^{B,R} &= \left(1 + i_{t(a)}\right)^{-1} \sum_{y=a}^{a^L} \left(b_{j,a}^R + \tilde{\beta}_{j,y,t(a)}^{S,R} + \beta_{y,t(a)}^S b_{j,a}^R\right) \frac{D_{y,t(a)}}{D_{a,t(a)}} \Leftrightarrow \\ \left(1 + i_{t(a)}\right) A_{j,a}^{B,R} &= \sum_{y=a}^{a^L} \tilde{\beta}_{j,y,t(a)}^{S,R} \frac{D_{y,t(a)}}{D_{a,t(a)}} + \sum_{y=a}^{a^L} \left(1 + \beta_{y,t(a)}^S\right) b_{j,a}^R \frac{D_{y,t(a)}}{D_{a,t(a)}} \end{aligned} \tag{6.109}$$

Using the definitions of $\varphi_{a,t}^{R,A}$ and $\tilde{\varphi}_{j,a,t}^R$ from above, we have

$$a_j^R < a \leq a^L : b_{j,a}^R = \varphi_{a,t(a)}^{R,A} \left(1 + i_{t(a)}\right) A_{j,a}^{B,R} - \tilde{\varphi}_{j,a,t(a)}^R \tag{6.110}$$

Finally, we define end-of period asset, $A^R_{j,a}$ for $a = a^R_j$ as

$$A^R_{j,a} = \left(1 + i_{t(a)}\right)\left(A^{B,R}_{j,a} + \tilde{A}^{B,R}_{j,a}\right) - \tilde{b}^R_{j,a} - b^R_{j,a} - q^{S,R}_{j,a} \Leftrightarrow \tag{6.111}$$

$$\left(1 + i_{t(a)}\right)\left(A^{B,R}_{j,a} + \left(1 - \tilde{d}^R_{j,a}\right)\tilde{A}^{B,R}_{j,a}\right) = A^R_{j,a} + \tilde{\beta}^{S,R}_{j,a,t(a)} + \left(1 + \beta^S_{a,t(a)}\right) b^R_{j,a} \tag{6.112}$$

where we have used eqs. (6.104) and (6.39). For $a > a^R_j$ we define end-of-period assets as

$$A^R_{j,a} = \left(1 + i_{t(a)}\right) A^{B,R}_{j,a} - b^R_{j,a} - q^{S,R}_{j,a} \Leftrightarrow \tag{6.113}$$

$$\left(1 + i_{t(a)}\right) A^{B,R}_{j,a} = A^R_{j,a} + \tilde{\beta}^{S,R}_{j,a,t(a)} + \left(1 + \beta^S_{a,t(a)}\right) b^R_{j,a} \tag{6.114}$$

where we have used eq. and (**??**)

Inserting eq. (6.112) in (6.105) or eq. (6.114) in (6.109) we get

$$A^R_{j,a} = \sum_{y=a+1}^{a^L} \tilde{\beta}^{S,R}_{j,y,t(a)} \frac{D_{y,t(a)}}{D_{a,t(a)}} + \sum_{y=a+1}^{a^L} \left(1 + \beta^S_{y,t(a)}\right) b^R_{j,a} \frac{D_{y,t(a)}}{D_{a,t(a)}} \tag{6.115}$$

This shows end-of-period assets to equal the discounted value of all forecasted future claims that the retired active member has on the pension fund.

6.4 The bonus

The derivation of pension undertakings, pensions and premiums uses the disablement and mortality rates entailed in the principles of precaution. As time evolves and disablement and death occur, a correction has to be made for two reasons:

- Actual disablement and mortality rates differ from those used in forecasting.
- Members are not identical with respect to contributions.

The correction takes the form of a percentage bonus on the assets of all surviving individuals and is calculated as follows.

Consider the members of age $a - 1$. At age $a - 1$ members have paid premiums to spouse pension and (if not retired) to disablement pension, which may be initiated at age a, and each individual (member or spouse pensioner) has end of period assets, $A_{j,a-1}$, as defined above.

Events of death and disablement are assumed to take place at the turn of each period. This means that for the individuals of age $a - 1$, death and disablement occur at the end of the period in which they are $a-1$ years old. Just after the turn of the period, that is at the beginning of the period in which the surviving individuals are a years old, we caculate the "required assets", $A^R_{j,a}$, for all *living* individuals, j, with a claim on the pension fund.

For a given individual the required assets are given by the discounted value of present (at age a) and forecasted future net claims of the individual based on contributions, pensions and pension undertakings of the *previous period*, i.e. based on the 'promises' that the pension fund made in the preceding period using the information of that period. Using the base interest rate these required assets are discounted one period further back to make them comparable with end of period assets at age $a-1$.

To be specific, consider a member j who became disabled at the end of period $a-1$. According to the calculations of the pension fund in period $t(a-1)$ this individual is now entitled to a pension with a present (at age a) value of

$$\tilde{b}^D_{j,a,t(a-1)} + \sum_{y=a}^{a^L} \tilde{\beta}^{S,D}_{j,y,t(a-1)} \frac{D_{y,t(a-1)}}{D_{a,t(a-1)}} + \sum_{y=a}^{a^L} \left(1+\beta^S_{y,t(a-1)}\right) w^D_{y,t(a-1)} b^U_{j,a-1} \frac{D_{y,t(a-1)}}{D_{a,t(a-1)}}$$

The first term is the once-and-for-all pension initiating disablement pension while the second two terms are the combined expenses to spouse pension insurance and the disablement pension. It should be noted that all technical terms, i.e. $\tilde{b}^D_{j,a,t(a-1)}$, $\tilde{\beta}^{S,D}_{j,y,t(a-1)}$, $\beta^S_{y,t(a-1)}$, $D_{y,t(a-1)}$ and $w^D_{y,t(a-1)}$ are those of period $t(a-1)$ because these are what were used in period $t(a-1)$ when the 'promise' of the disablement pension was made by the pension fund. Therefore, also the life-long disablement pension is initiated using the pension undertaking of the previous period, $b^U_{j,a-1}$.

We now discount the required assets at age a to period $t(a-1)$ to make them comparable with end-of-period assets of that period. This gives required assets (we use the base interest rate in period $t(a-1)$ because this is what was used when making the calculations in period $t(a-1)$)

$$\begin{aligned} A^{R,D}_{j,a,a} &= \frac{1}{1+\bar{\imath}_{t(a-1)}} \\ &\times \left[\tilde{b}^D_{j,a,t(a-1)} + \sum_{y=a}^{a^L} \tilde{\beta}^{S,D}_{j,y,t(a-1)} \frac{D_{y,t(a-1)}}{D_{a,t(a-1)}} + \sum_{y=a}^{a^L} \left(1+\beta^S_{y,t(a-1)}\right) w^D_{y,t(a-1)} b^U_{j,a-1} \frac{D_{y,t(a-1)}}{D_{a,t(a-1)}}\right] \end{aligned}$$

Comparison with eq. (6.61) shows that

$$A^{R,D}_{j,a,a} = \frac{1}{\bar{r}^D_{a,t(a-1)} \left(1 - r^M_{a,t(a-1)}\right)} q^D_{j,a-1} \tag{6.116}$$

If we now instead turn to an on-going disablement pension the present value at age a of the required assets to meet the calculated (as of age $a-1$) disablement pension and spouse pension insurance coverage is given by

$$\sum_{y=a}^{a^L} \tilde{\beta}^{S,D}_{j,y,t(a-1)} \frac{D_{y,t(a-1)}}{D_{a,t(a-1)}} + \sum_{y=a}^{a^L} \left(1+\beta^S_{y,t(a-1)}\right) b^D_{j,a-1,d} \frac{D_{y,t(a-1)}}{D_{a,t(a-1)}}$$

Discounting to the previous period and making use of eq. (6.20) we now get

$$\begin{aligned}
& A^{R,D}_{j,a,d} \\
= \; & \frac{1}{1+\bar{\imath}_{t(a-1)}} \left(\sum_{y=a}^{a^L} \tilde{\beta}^{S,D}_{j,y,t(a-1)} \frac{D_{y,t(a-1)}}{D_{a,t(a-1)}} + \sum_{y=a}^{a^L} \left(1+\beta^S_{y,t(a-1)}\right) b^D_{j,a-1,d} \frac{D_{y,t(a-1)}}{D_{a,t(a-1)}} \right) \\
= \; & \frac{1}{1+\bar{\imath}_{t(a-1)}} \sum_{y=a}^{a^L} \tilde{\beta}^{S,D}_{j,y,t(a-1)} \frac{1+\bar{\imath}_{t(a-1)}}{1-\bar{r}^M_{a,t(a-1)}} \frac{D_{y,t(a-1)}}{D_{a-a,t(a-1)}} \\
& + \frac{1}{1+\bar{\imath}_{t(a-1)}} \sum_{y=a}^{a^L} \left(1+\beta^S_{y,t(a-1)}\right) b^D_{j,a-1,d} \frac{1+\bar{\imath}_{t(a-1)}}{1-\bar{r}^M_{a,t(a-1)}} \frac{D_{y,t(a-1)}}{D_{a-1,t(a-1)}} \\
= \; & \frac{1}{1-\bar{r}^M_{a,t(a-1)}} \left(\sum_{y=a}^{a^L} \tilde{\beta}^{S,D}_{j,y,t(a-1)} \frac{D_{y,t(a-1)}}{D_{a-a,t(a-1)}} + \sum_{y=a}^{a^L} \left(1+\beta^S_{y,t(a-1)}\right) b^D_{j,a-1,d} \frac{D_{y,t(a-1)}}{D_{a-1,t(a-1)}} \right) \\
= \; & \frac{1}{1-\bar{r}^M_{a,t(a-1)}} A^D_{j,a-1,d} \qquad (6.117)
\end{aligned}$$

by comparison with eq. (6.75).

MERE OM FORTOLKNINGNERN

Equations (6.116) and (6.117) very clearly shows the basic insurance principle behind the pension fund, which is to pool the risc of death and disablement among the members. We focus on eq. (6.116) letting N^N_{a-1} be the number of active, non-retired individuals of age $a-1$. If we assume that actual disablement occurs according to the probabilities of the technical basis in period $t(a-1)$, then the number of newly disabled pensioners of age a in period $t(a)$ will be

$$N^D_{a,a} = \bar{r}^D_{a,t(a-1)} \left(1-\bar{r}^M_{a,t(a-1)}\right) N^N_{a-1} \qquad (6.118)$$

The total value of premiums paid be the active members at age $a-1$ will be (assuming all individuals are identical)

$$N^N_{a-1} q^D_{j,a-1}$$

while the required assets (measured in present value in period $t(a-1)$) to meet the claims of the newly disabled at age a are (once again assuming that all individuals are identical)

$$\begin{aligned} N^D_{a,a} A^{R,D}_{j,a,a} &= \bar{r}^D_{a,t(a-1)} \left(1 - \bar{r}^M_{a,t(a-1)}\right) N^N_{a-1} \frac{1}{\bar{r}^D_{a,t(a-1)} \left(1 - r^M_{a,t(a-1)}\right)} q^D_{j,a-1} \\ &= N^N_{a-1} q^D_{j,a-1} \end{aligned}$$

I.e. the total sum of required assets is exactly equal to the sum of premiums paid in the preceding period. In a similar manner it may be shown that the total sum of the required assets for ongoing disablement pension in eq. (6.117) for those who survive is exactly equal to the sum of end-of-period-assets for disablement pensioners of the preceding period. It should be emphasized that this result only holds if forecasted probabilities are correct and if all members are exactly identical (and if the law of large numbers can be invoked). Because this can not be expected to hold generally a bonus is needed. This bonus takes the form of a percentage addition to the required assets in a given period, which ensures that the total sum of end-of-period assets of the preceding period plus all premiums paid in that period should equal the total sum of member's assets at the beginning of that period. To be specific, the bonus of period t is calculated as:

$$bonus_t = \frac{\sum_{\text{Individuals in period } t-1} (\text{End of period } t-1 \text{ assets} + \text{Premiums paid in period } t-1)}{\sum_{\text{Individuals in period } t} \text{Required assets at beginning of period } t} - 1$$

This bonus is then given to all individuals with a claim on the pension fund at the beginning of period t by setting the beginning of period assets $A^2_{j,y}$ which are used above for all types of pensions equal to required assets including bonus, i.e.

$$A^B_{j,a} = \left(1 + bonus_{t(a)}\right) A^R_{j,a} \tag{6.119}$$

Thus it is assured that the sum of all $A^B_{j,a}$ is equal to the sum of all end-of-period asset $A_{j,a-1}$ plus all premiums paid in period $t(a-1)$.

6.4.1 Overview of required assets

This section gives all required assets, which are found in a way similar to those of the preceding section and with all derivations made formally in the appendix.

Non-retired members

Her skal også være noget, hvor vi definerer $A_{j,y}^{1,ZF,U,Once}$ for de nyligt pensionerede, altså nomenklaturmæssigt $A_{j,y}^{1,ZF,R,Once}$

$$a = a_j^F : A_{j,y}^{R,N} = 0 \tag{6.120}$$

$$a \leq a^R \wedge a \leq a_j^R : A_{j,y}^{R,N} = \frac{1}{\left(1-\bar{r}_{a,t(a-1)}^D\right)\left(1-\bar{r}_{a,t(a-1)}^M\right)} A_{j,a-1}^N \tag{6.121}$$

$$a^R < a < a_j^R : A_{j,a}^{R,N} = \frac{1}{1-\bar{r}_{a,t(a-1)}^M} A_{j,a-1}^N \tag{6.122}$$

$$a = a_j^F : \tilde{A}_{j,a}^{R,N} = 0 \tag{6.123}$$

$$a \leq a^R \wedge a < a_j^R : \tilde{A}_{j,a}^{R,N} = \frac{1}{\left(1-\bar{r}_{a,t(a-1)}^D\right)\left(1-\bar{r}_{a,t(a-1)}^M\right)} \tilde{A}_{j,a-1}^N \tag{6.124}$$

$$a^R < a < a_j^R : \tilde{A}_{j,a}^{R,N} = \frac{1}{1-\bar{r}_{a,t(a-1)}^M} \tilde{A}_{j,a-1}^N \tag{6.125}$$

Active retired members

$$a = a_j^R \leq a^R : \tilde{A}_{j,a}^{R,R} = \frac{1}{\left(1-\bar{r}_{a,t(a-1)}^D\right)\left(1-\bar{r}_{a,t(a-1)}^M\right)} \tilde{A}_{j,a-1}^N \tag{6.126}$$

$$a = a_j^R > a^R : \tilde{A}_{j,a}^{R,R} = \frac{1}{1-\bar{r}_{a,t(a-1)}^M} \tilde{A}_{j,a-1}^N \tag{6.127}$$

$$a > a_j^R : \tilde{A}_{j,a}^{R,R} = 0 \tag{6.128}$$

$$a = a_j^R \leq a^R : A_{j,a}^{R,R} = \frac{1}{\left(1-\bar{r}_{a,t(a-1)}^D\right)\left(1-\bar{r}_{a,t(a-1)}^M\right)} A_{j,a-1}^N \tag{6.129}$$

$$a = a_j^R > a^R : A_{j,a}^{R,R} = \frac{1}{1-\bar{r}_{a,t(a-1)}^M} A_{j,a-1}^N \tag{6.130}$$

$$a > a_j^R : A_{j,a}^{R,R} = \frac{1}{1-\bar{r}_{a,t(a-1)}^M} A_{j,a-1}^R \tag{6.131}$$

Spouse pensioners

$$A_{j,a,a}^{R,S,N} = \frac{1}{r_{a,t(a-1)}^M \bar{r}_{a,t(a-1)}^S} q_{\hat{j},a-1}^{S,N} \tag{6.132}$$

$$A_{j,a,s}^{R,S,N} = \frac{1}{1-\bar{r}_{a,t(a-1)}^M} A_{j,a-1,s}^{S,N}, \quad a > s \tag{6.133}$$

$$A_{j,a,a}^{R,S,R} = \frac{1}{r_{a,t(a-1)}^M \bar{r}_{a,t(a-1)}^S} q_{\hat{j},a-1}^{S,R} \tag{6.134}$$

$$A^{R,S,R}_{j,a,s} = \frac{1}{1 - \bar{r}^{M}_{a,t(a-1)}} A^{S,R}_{j,a-1,s}, \quad a > s \tag{6.135}$$

$$A^{R,S,D}_{j,a,a,d} = \frac{1}{\bar{r}^{M}_{a,t(a-1)} r^{S}_{a,t(a-1)}} q^{S,D}_{\hat{j},a-1,d} \tag{6.136}$$

$$A^{R,S,D}_{j,a,s,d} = \frac{1}{1 - \bar{r}^{M}_{a,t(a-1)}} A^{S,D}_{j,a-1,s,d}, \quad a > s > d \tag{6.137}$$

Disablement pensioners

$$A^{R,D}_{j,a,a} = \frac{1}{\bar{r}^{D}_{a,t(a-1)} \left(1 - \bar{r}^{M}_{a,t(a-1)}\right)} q^{D}_{j,a-1} \tag{6.138}$$

$$A^{R,D}_{j,a,d} = \frac{1}{1 - \bar{r}^{M}_{a,t(a-1)}} A^{D}_{j,a-1,d}, \quad a > d \tag{6.139}$$

6.5 Evolution of pensions

Heptjulej - en indlende tekst.

By using (6.110) we obtain

$$\begin{aligned}
b^R_{j,a} &= \varphi^{R,A}_{a,t(a)}\left(1+i_{t(a)}\right)A^{B,R}_{j,a} - \tilde{\varphi}^R_{j,a,t(a)} \\
\text{(6.119)} &= \varphi^{R,A}_{a,t(a)}\left(1+i_{t(a)}\right)\left(1+bonus_{t(a)}\right)A^{R,R}_{j,a} - \tilde{\varphi}^R_{j,a,t(a)} \\
\text{(??)} &= \varphi^{R,A}_{a,t(a)}\left(1+i_{t(a)}\right)\left(1+bonus_{t(a)}\right)\frac{1}{1-\bar{r}^M_{a,t(a-1)}}A^R_{j,a-1} - \tilde{\varphi}^R_{j,a,t(a)} \\
\text{(6.115)} &= \varphi^{R,A}_{a,t(a)}\left(1+i_{t(a)}\right)\left(1+bonus_{t(a)}\right)\frac{1}{1-\bar{r}^M_{a,t(a-1)}} \\
&\quad \times\left(\sum_{y=a}^{a^L}\tilde{\beta}^{S,R}_{j,y,t(a-1)}\frac{D_{y,t(a-1)}}{D_{a-1,t(a-1)}} + \sum_{y=a}^{a^L}\left(1+\beta^S_{y,t(a-1)}\right)b^R_{j,a-1}\frac{D_{y,t(a-1)}}{D_{a,t(a-1)}}\right) \\
&\quad -\tilde{\varphi}^R_{j,a,t(a)} \\
\text{(6.19)} &= \varphi^{R,A}_{a,t(a)}\left(1+i_{t(a)}\right)\left(1+bonus_{t(a)}\right)\frac{1}{1-\bar{r}^M_{a,t(a-1)}} \\
&\quad \times\left(\sum_{y=a}^{a^L}\tilde{\beta}^{S,R}_{j,y,t(a-1)}\frac{1-\bar{r}^M_{a,t(a-1)}}{1+\bar{\imath}_{t(a-1)}}\frac{D_{y,t(a-1)}}{D_{a,t(a-1)}} + \sum_{y=a}^{a^L}\left(1+\beta^S_{y,t(a-1)}\right)b^R_{j,a-1}\frac{1-\bar{r}^M_{a,t(a-1)}}{1+\bar{\imath}_{t(a-1)}}\frac{D_{y,t(a-1)}}{D_{a,t(a-1)}}\right) \\
&\quad -\tilde{\varphi}^R_{j,a,t(a)} \\
&= \varphi^{R,A}_{a,t(a)}\frac{1+i_{t(a)}}{1+\bar{\imath}_{t(a-1)}}\left(1+bonus_{t(a)}\right) \\
&\quad \times\left(\sum_{y=a}^{a^L}\tilde{\beta}^{S,R}_{j,y,t(a-1)}\frac{D_{y,t(a-1)}}{D_{a,t(a-1)}} + \sum_{y=a}^{a^L}\left(1+\beta^S_{y,t(a-1)}\right)b^R_{j,a-1}\frac{D_{y,t(a-1)}}{D_{a,t(a-1)}}\right) \\
&\quad -\tilde{\varphi}^R_{j,a,t(a)} \\
&= \varphi^{R,A}_{a,t(a)}\frac{1+i_{t(a)}}{1+\bar{\imath}_{t(a-1)}}\left(1+bonus_{t(a)}\right) \\
&\quad \times\left(\frac{\tilde{\varphi}^R_{j,a,t(a-1)}}{\varphi^{R,A}_{a,t(a-1)}} + \frac{1}{\varphi^{R,A}_{a,t(a-1)}}b^R_{j,a-1}\right) \\
&\quad -\tilde{\varphi}^R_{j,a,t(a)} \\
&= \frac{\varphi^{R,A}_{a,t(a)}}{\varphi^{R,A}_{a,t(a-1)}}\frac{1+i_{t(a)}}{1+\bar{\imath}_{t(a-1)}}\left(1+bonus_{t(a)}\right)b^R_{j,a-1} + \frac{\varphi^{R,A}_{a,t(a)}}{\varphi^{R,A}_{a,t(a-1)}}\frac{1+i_{t(a)}}{1+\bar{\imath}_{t(a-1)}}\left(1+bonus_{t(a)}\right)\tilde{\varphi}^R_{j,a,t(a-1)}.
\end{aligned}$$

Assuming that there is no one time spouse pension, i.e. assuming $\tilde{\varphi}^R_{j,a,t} = 0$ we get

$$b^R_{j,a} = \frac{\varphi^{R,A}_{a,t(a)}}{\varphi^{R,A}_{a,t(a-1)}}\frac{1+i_{t(a)}}{1+\bar{\imath}_{t(a-1)}}\left(1+bonus_{t(a)}\right)b^R_{j,a-1}$$

Furthermore, assuming no change in the technical basis from period $t(a-1)$ to period $t(a)$ we have

$$b^R_{j,a} = \frac{1+i_{t(a)}}{1+\bar{\imath}_{t(a-1)}}\left(1+bonus_{t(a)}\right)b^R_{j,a-1}$$

This shows the ...

6.6 Average pensions

Om j og k.

6.6.1 Non-retired members

For any variable, $X^N_{k,a,e,t}$, related to non-retired members, we define the average value as

$$X^N_{k,a,t} \equiv \frac{\sum_{e=a^F}^{a} N^N_{k,a,e,t} X^N_{k,a,e,t}}{N^N_{k,a,t}}$$

The average relations are

$$\begin{aligned}
a^F \le a < a^{LR}: &\quad q^{S,N}_{k,a,t} = \tilde{\beta}^{S,N}_{k,a,t} + \beta^S_{a,t} b^U_{k,a,t} \\
a^F \le a < a^R: &\quad q^D_{k,a,t} = \tilde{\beta}^D_{k,a,t} + \beta^D_{a,t} b^U_{k,a,t} \\
a^F \le a < a^{LR}: &\quad b^U_{k,a,t} = \varphi^{U,A}_{a,t} (1+i_t) \left(A^{B,N}_{k,a,t} + \tilde{A}^{B,N}_{k,a,t} \right) + \varphi^{U,q}_{a,t} q_{k,a,t} - \tilde{\varphi}^U_{k,a,t} \\
a^F \le a < a^R: &\quad A^N_{k,a,t} = (1+i_t) \left(A^{B,N}_{k,a,t} + \tilde{A}^{B,N}_{k,a,t} \right) + q_{k,a,t} - q^D_{k,a,t} - q^{S,N}_{k,a,t} - \tilde{A}^N_{k,a,t} \\
a^R \le a < a^{LR}: &\quad A^N_{k,a,t} = (1+i_t) \left(A^{B,N}_{k,a,t} + \tilde{A}^{B,N}_{k,a,t} \right) + q_{k,a,t} - q^{S,N}_{k,a,t} - \tilde{A}^N_{k,a,t} \\
a^F \le a < a^{LR}: &\quad \tilde{A}^N_{k,a,t} = (1+i_t) \tilde{A}^{B,N}_{k,a,t} + \tilde{q}_{k,a,t} \\
a^F \le a < a^{LR}: &\quad A^{B,N}_{k,a,t} = (1+bonus_t) A^{R,N}_{k,a,t} \\
a^F \le a < a^{LR}: &\quad \tilde{A}^{B,N}_{k,a,t} = (1+bonus_t) \tilde{A}^{R,N}_{k,a,t} \\
a = a^F: &\quad A^{R,N}_{k,a,t} = 0 \\
a^F < a < a^{FR}: &\quad A^{R,N}_{k,a,t} = \frac{1}{\left(1-\bar{r}^D_{a,t-1}\right)\left(1-\bar{r}^M_{a,t-1}\right)} A^N_{k,a-1,t-1} \left(1 - \frac{N^{N,Init}_{k,a,t}}{N^N_{k,a,t}} \right) \\
a^{FR} \le a \le a^R: &\quad A^{R,N}_{k,a,t} = \frac{1}{\left(1-\bar{r}^D_{a,t-1}\right)\left(1-\bar{r}^M_{a,t-1}\right)} A^N_{k,a-1,t-1} \\
a^R < a < a^{LR}: &\quad A^{R,N}_{k,a,t} = \frac{1}{\left(1-\bar{r}^M_{a,t-1}\right)} A^N_{k,a-1,t-1} \\
a = a^F: &\quad \tilde{A}^{R,N}_{k,a,t} = 0 \\
a^F < a < a^{FR}: &\quad \tilde{A}^{R,N}_{k,a,t} = \frac{1}{\left(1-\bar{r}^D_{a,t-1}\right)\left(1-\bar{r}^M_{a,t-1}\right)} \tilde{A}^N_{k,a-1,t-1} \left(1 - \frac{N^{N,Init}_{k,a,t}}{N^N_{k,a,t}} \right) \\
a^{FR} \le a \le a^R: &\quad \tilde{A}^{R,N}_{k,a,t} = \frac{1}{\left(1-\bar{r}^D_{a,t-1}\right)\left(1-\bar{r}^M_{a,t-1}\right)} \tilde{A}^N_{k,a-1,t-1} \\
a^R < a < a^{LR}: &\quad \tilde{A}^{R,N}_{k,a,t} = \frac{1}{\left(1-\bar{r}^M_{a,t-1}\right)} \tilde{A}^N_{k,a-1,t-1}
\end{aligned}$$

6.6.2 Retired members

$$X_{k,a,t} \equiv \frac{\sum_{r=a^{FR}}^{a} \sum_{e=a^F}^{a^{FR}-1} N^R_{k,a,r,e,t} X_{k,a,r,e,t}}{N^R_{k,a,t}}$$

NB $\tilde{d}^R_{k,a,t}$ er ikke et gennemsnit; det antages, at den er den samme for alle.

$$\begin{aligned}
a^{FR} \le a \le a^L : &\quad q^{S,R}_{k,a,t} = \tilde{\beta}^{S,R}_{k,a,t} + \beta^S_{a,t} b^R_{k,a,t} \\
a^{FR} \le a \le a^{LR} : &\quad b^R_{k,a,t} = \varphi^{R,A}_{a,t} (1+i_t) \left(A^{B,R}_{k,a,t} + \left(1 - \tilde{d}^R_{k,a,t}\right) \tilde{A}^{B,R}_{k,a,t} \right) - \tilde{\varphi}^R_{k,a,t} \\
a^{LR} < a \le a^L : &\quad b^R_{k,a,t} = \varphi^{R,A}_{a,t} (1+i_t) A^{B,R}_{k,a,t} - \tilde{\varphi}^R_{k,a,t} \\
a^{FR} \le a \le a^{LR} : &\quad \tilde{b}^R_{k,a,t} = (1+i_t) \tilde{d}_{k,a,t} \tilde{A}^{B,R}_{k,a,t} \\
a^{FR} \le a \le a^{LR} : &\quad A^R_{k,a,t} = (1+i_t) \left(A^{B,R}_{k,a,t} + \tilde{A}^{B,R}_{k,a,t} \right) - \tilde{b}^R_{k,a,t} - b^R_{k,a,t} - q^{S,R}_{k,a,t} \\
a^{FR} < a \le a^L : &\quad A^R_{k,a,t} = (1+i_t) A^{B,R}_{k,a,t} - b^R_{k,a,t} - q^{S,R}_{k,a,t} \\
a^{FR} \le a \le a^L : &\quad A^{B,R}_{k,a,t} = (1+bonus_t) A^{R,R}_{k,a,t} \\
a^{FR} \le a \le a^{LR} : &\quad \tilde{A}^{B,R}_{k,a,t} = (1+bonus_t) \tilde{A}^{R,R}_{k,a,t} \\
a = a^{FR} : &\quad A^{R,R}_{k,a,t} = \frac{1}{\left(1-\bar{r}^D_{a,t}\right)\left(1-\bar{r}^M_{a,t}\right)} A^N_{k,a-1,t-1} \\
a^{FR} < a \le a^R : &\quad A^{R,R}_{k,a,t} = \frac{1}{1-\bar{r}^M_{a,t}} A^R_{k,a-1,t-1} \left(1 - \frac{N^{R,Init}_{k,a,t}}{N^R_{k,a,t}}\right) + \frac{1}{\left(1-\bar{r}^D_{a,t}\right)\left(1-\bar{r}^M_{a,t}\right)} A^N_{k,a-1,t-1} \frac{N^{R,Ini}_{k,a,t}}{N^R_{k,a,t}} \\
a^R < a \le a^{LR} : &\quad A^{R,R}_{k,a,t} = \frac{1}{1-\bar{r}^M_{a,t}} A^R_{k,a-1,t-1} \left(1 - \frac{N^{R,Init}_{k,a,t}}{N^R_{k,a,t}}\right) + \frac{1}{\left(1-\bar{r}^M_{a,t}\right)} A^N_{k,a-1,t-1} \frac{N^{R,Init}_{k,a,t}}{N^R_{k,a,t}} \\
a^{LR} < a \le a^L : &\quad A^{R,R}_{k,a,t} = \frac{1}{1-\bar{r}^M_{a,t}} A^R_{k,a-1,t-1} \\
a = a^{FR} : &\quad \tilde{A}^{R,R}_{k,a,t} = \frac{1}{\left(1-\bar{r}^D_{a,t}\right)\left(1-\bar{r}^M_{a,t}\right)} \tilde{A}^N_{k,a-1,t-1} \\
a^{FR} < a \le a^R : &\quad \tilde{A}^{R,R}_{k,a,t} = \frac{1}{\left(1-\bar{r}^D_{a,t}\right)\left(1-\bar{r}^M_{a,t}\right)} \tilde{A}^N_{k,a-1,t-1} \frac{N^{R,Init}_{k,a,t}}{N^R_{k,a,t}} \\
a^R < a \le a^{LR} : &\quad \tilde{A}^{R,R}_{k,a,t} = \frac{1}{\left(1-\bar{r}^M_{a,t}\right)} \tilde{A}^N_{k,a-1,t-1} \frac{N^{R,Init}_{k,a,t}}{N^R_{k,a,t}}
\end{aligned}$$

6.6.3 Disabled members

For any variable, $X^D_{k,a,d,e,t}$, related to disabled members, we define the average value as

$$X^D_{k,a,t} \equiv \frac{\sum_{d=a^F+1}^{a} \sum_{e=a^F}^{a-1} N^D_{k,a,d,e,t} X_{k,a,d,e,t}}{N^D_{k,a,t}}$$

except for $\tilde{b}^D_{k,a,t}$, which is equal to $\tilde{b}^D_{j(k),a,t}$, since it is independant of e in the first place.

$$
\begin{aligned}
a^F < a \leq a^L: &\quad q^{S,D}_{k,a,t} = \tilde{\beta}^{S,D}_{k,a,t} + \beta^S_{a,t} b^D_{k,a,t} \\
a = a^F + 1: &\quad b^D_{k,a,t} = \varphi^D_{a,t}\left((1+i_t)\, A^{B,D}_{k,a,t} - \tilde{b}^D_{k,a,t}\right) - \tilde{\varphi}^D_{k,a,t} \\
a^F + 1 < a \leq a^R: &\quad b^D_{k,a,t} = \varphi^D_{a,t}\left((1+i_t)\, A^{B,D}_{k,a,t} - \tilde{b}^D_{k,a,t}\frac{N^{D,Init}_{k,a,t}}{N^D_{k,a,t}}\right) - \tilde{\varphi}^D_{k,a,t} \\
a^R < a \leq a^{LR}: &\quad b^D_{k,a,t} = \varphi^D_{a,t}\,(1+i_t)\, A^{B,D}_{k,a,t} - \tilde{\varphi}^D_{k,a,t} \\
a = a^F + 1: &\quad A^D_{k,a,t} = (1+i_t)\, A^{B,D}_{k,a,t} - b^D_{k,a,t} - \tilde{b}^D_{k,a,t} - q^{S,D}_{k,a,t} \\
a^F + 1 < a \leq a^R: &\quad A^D_{k,a,t} = (1+i_t)\, A^{B,D}_{k,a,t} - b^D_{k,a,t} - \tilde{b}^D_{k,a,t}\frac{N^{D,Init}_{k,a,t}}{N^D_{k,a,t}} - q^{S,D}_{k,a,t} \\
a^R < a \leq a^{LR}: &\quad A^D_{k,a,t} = (1+i_t)\, A^{B,D}_{k,a,t} - b^D_{k,a,t} - q^{S,D}_{k,a,t} \\
a^F < a < a^{LR}: &\quad A^{B,D}_{k,a,t} = (1+bonus_t)\, A^{R,D}_{k,a,t} \\
a = a^F + 1: &\quad A^{R,D}_{k,a,t} = \frac{1}{\bar{r}^D_{a,t-1}\left(1-\bar{r}^M_{a,t-1}\right)} q^D_{k,a,t} \\
a^F + 1 < a \leq a^R: &\quad A^{R,D}_{k,a,t} = \frac{1}{\bar{r}^D_{a,t-1}\left(1-\bar{r}^M_{a,t-1}\right)} q^D_{k,a,t}\frac{N^{D,Init}_{k,a,t}}{N^D_{k,a,t}} + \frac{1}{\left(1-\bar{r}^M_{a,t-1}\right)} A^R_{k,a-1,t-1}\left(1 - \frac{N^{D,Init}_{k,a,t}}{N^D_{k,a,t}}\right) \\
a^R < a \leq a^L: &\quad A^{R,D}_{k,a,t} = \frac{1}{\left(1-\bar{r}^M_{a,t-1}\right)} A^R_{k,a-1,t-1}\left(1 - \frac{N^{D,Init}_{k,a,t}}{N^D_{k,a,t}}\right)
\end{aligned}
$$

6.6.4 Spouse pensioners

We define

$$
\begin{aligned}
X^S_{k,a,t} &= \frac{1}{N^S_{k,a,t}} \\
\times &\quad \left[\sum_{s=a^F+1}^{a}\sum_{e=a^F}^{s-1} N^{S,N}_{k,a,s,e,t} X^{S,N}_{k,a,s,e,t}\right. \\
&\quad + \sum_{s=a^F+2}^{a}\sum_{d=a^F+1}^{s-1}\sum_{e=a^F}^{d-1} N^{S,D}_{k,a,s,d,e,t} X^{S,D}_{k,a,s,d,e,t} \\
&\quad \left. + \sum_{s=a^{FR}+1}^{a}\sum_{r=a^{FR}}^{s-1}\sum_{e=a^F}^{a^{FR}-1} N^{S,R}_{k,a,s,r,e,t} X^{S,R}_{k,a,s,r,e,t}\right]
\end{aligned}
$$

$$a = a^F + 1: \quad b^S_{k,a,t} = \varphi^S_{a,t} \left((1 + i_t)\, A^{B,S}_{k,a,t} - \tilde{b}^{S,N}_{k,a,t} \right)$$

$$a^F + 1 < a \leq a^{FR}: \quad b^S_{k,a,t} = \varphi^S_{a,t} \left((1 + i_t)\, A^{B,S}_{k,a,t} - \tilde{b}^{S,N}_{k,a,t} \frac{N^{S,N,Init}_{k,a,t}}{N^S_{k,a,t}} - \tilde{b}^{S,D}_{k,a,t} \frac{N^{S,D,Init}_{k,a,t}}{N^S_{k,a,t}} \right)$$

$$a^{FR} < a \leq a^{LR}: \quad b^S_{k,a,t} = \varphi^S_{a,t} \left((1 + i_t)\, A^{B,S}_{k,a,t} - \tilde{b}^{S,N}_{k,a,t} \frac{N^{S,N,Init}_{k,a,t}}{N^S_{k,a,t}} - \tilde{b}^{S,D}_{k,a,t} \frac{N^{S,D,Init}_{k,a,t}}{N^S_{k,a,t}} - \tilde{b}^{S,R}_{k,a,t} \frac{N^{S,R,Init}_{k,a,t}}{N^S_{k,a,t}} \right)$$

$$a^{LR} < a \leq a^L: \quad b^S_{k,a,t} = \varphi^S_{a,t} \left((1 + i_t)\, A^{B,S}_{k,a,t} - \tilde{b}^{S,D}_{k,a,t} \frac{N^{S,D,Init}_{k,a,t}}{N^S_{k,a,t}} - \tilde{b}^{S,R}_{k,a,t} \frac{N^{S,R,Init}_{k,a,t}}{N^S_{k,a,t}} \right)$$

$$a = a^F + 1: \quad A^S_{k,a,t} = (1 + i_t)\, A^{B,S}_{k,a,t} - b^S_{k,a,t} - \tilde{b}^{S,N}_{k,a,t}$$

$$a^F + 1 < a \leq a^{FR}: \quad A^S_{k,a,t} = (1 + i_t)\, A^{B,S}_{k,a,t} - b^S_{k,a,t} - \tilde{b}^{S,N}_{k,a,t} \frac{N^{S,N,Init}_{k,a,t}}{N^S_{k,a,t}} - \tilde{b}^{S,D}_{k,a,t} \frac{N^{S,D,Init}_{k,a,t}}{N^S_{k,a,t}}$$

$$a^{FR} < a \leq a^{LR}: \quad A^S_{k,a,t} = (1 + i_t)\, A^{B,S}_{k,a,t} - b^S_{k,a,t} - \tilde{b}^{S,N}_{k,a,t} \frac{N^{S,N,Init}_{k,a,t}}{N^S_{k,a,t}} - \tilde{b}^{S,D}_{k,a,t} \frac{N^{S,D,Init}_{k,a,t}}{N^S_{k,a,t}} - \tilde{b}^{S,R}_{k,a,t} \frac{N^{S,R,Ini}_{k,a,t}}{N^S_{k,a,t}}$$

$$a^{LR} < a \leq a^L: \quad A^S_{k,a,t} = (1 + i_t)\, A^{B,S}_{k,a,t} - b^S_{k,a,t} - \tilde{b}^{S,D}_{k,a,t} \frac{N^{S,D,Init}_{k,a,t}}{N^S_{k,a,t}} - \tilde{b}^{S,R}_{k,a,t} \frac{N^{S,R,Init}_{k,a,t}}{N^S_{k,a,t}}$$

$$a^F < a \leq a^L: \quad A^{B,S}_{k,a,t} = (1 + bonus_t)\, A^{R,S}_{k,a,t}$$

$$a = a^F + 1: \quad A^{R,S}_{k,a,t} = \frac{1}{r^M_{a,t-1} \bar{r}^S_{a,t-1}} q^{S,N}_{\hat{k},a-1,t-1}$$

$$\begin{aligned} a^F + 1 < a \leq a^{FR}: \quad A^{R,S}_{k,a,t} &= \frac{1}{1 - \bar{r}^M_{a,t-1}} A^{S,N}_{k,a-1,t-1} \left(1 - \frac{N^{S,N,Init}_{k,a,t}}{N^S_{k,a,t}} - \frac{N^{S,D,Init}_{k,a,t}}{N^S_{k,a,t}} \right) \\ &+ \frac{1}{r^M_{a,t-1} \bar{r}^S_{a,t-1}} q^{S,N}_{\hat{k},a-1,t-1} \frac{N^{S,N,Init}_{k,a,t}}{N^S_{k,a,t}} \\ &+ \frac{1}{r^M_{a,t-1} \bar{r}^S_{a,t-1}} q^{S,D}_{\hat{k},a-1,t-1} \frac{N^{S,D,Init}_{k,a,t}}{N^S_{k,a,t}} \end{aligned}$$

$$\begin{aligned} a^{FR} < a \leq a^{LR}: \quad A^{R,S}_{k,a,t} &= \frac{1}{1 - \bar{r}^M_{a,t-1}} A^{S,N}_{k,a-1,t-1} \left(1 - \frac{N^{S,N,Init}_{k,a,t}}{N^S_{k,a,t}} - \frac{N^{S,D,Init}_{k,a,t}}{N^S_{k,a,t}} \right) \\ &+ \frac{1}{r^M_{a,t-1} \bar{r}^S_{a,t-1}} q^{S,N}_{\hat{k},a-1,t-1} \frac{N^{S,N,Init}_{k,a,t}}{N^S_{k,a,t}} \\ &+ \frac{1}{r^M_{a,t-1} \bar{r}^S_{a,t-1}} q^{S,D}_{\hat{k},a-1,t-1} \frac{N^{S,D,Init}_{k,a,t}}{N^S_{k,a,t}} \\ &+ \frac{1}{r^M_{a,t-1} \bar{r}^S_{a,t-1}} q^{S,R}_{\hat{k},a-1,t-1} \frac{N^{S,R,Init}_{k,a,t}}{N^S_{k,a,t}} \end{aligned}$$

$$\begin{aligned} a^{LR} < a \leq a^L: \quad A^{R,S}_{k,a,t} &= \frac{1}{1 - \bar{r}^M_{a,t-1}} A^{S,N}_{k,a-1,t-1} \left(1 - \frac{N^{S,N,Init}_{k,a,t}}{N^S_{k,a,t}} - \frac{N^{S,D,Init}_{k,a,t}}{N^S_{k,a,t}} \right) \\ &+ \frac{1}{r^M_{a,t-1} \bar{r}^S_{a,t-1}} q^{S,D}_{\hat{k},a-1,t-1} \frac{N^{S,D,Init}_{k,a,t}}{N^S_{k,a,t}} \\ &+ \frac{1}{r^M_{a,t-1} \bar{r}^S_{a,t-1}} q^{S,R}_{\hat{k},a-1,t-1} \frac{N^{S,R,Init}_{k,a,t}}{N^S_{k,a,t}} \end{aligned}$$

6.6.5 Member stocks

Non-retired members

New members enter into the pension fund over time and since accumulated assets and thus pension undertakings and so forth depend on how long they have been members we need to keep track of the time of entry into the pension fund. Let $N^N_{k,a,e,t}$ be the total number of members of category k and age a in period t who entered at age $e \leq a$. The total stock of active, non-retired members of category k and age a in period t is therefore

$$N^N_{k,a,t} \equiv \sum_{e=a^F}^{a} N^N_{k,a,e,t} \tag{6.140}$$

Because no new members enter the pension fund after age $a^{FR}-1$ the definition in eq. (6.140) may be written as

$$N^N_{k,a,t} \equiv \sum_{e=a^F}^{a^{FR}-1} N^N_{k,a,e,t} \tag{6.141}$$

for $a \geq a^{FR}$.

For later use we invoke the following notation

$$N^{N,Init}_{k,a,t} = N^N_{k,a,a,t}$$

with

$$N^{N,Init}_{k,a,t} = N^N_{k,a,a,t} = 0, \quad a \geq a^{FR} \tag{6.142}$$

We let $r^M_{k,a,t}$ be the fraction of the stock of members of age $a-1$ who are no longer members at age a, either because they have died or because they have left the country.NOTE om emigration. Similarly we let $r^D_{k,a,t}$ be the fraction of non-retired members at age $a-1$ who are disabled and who experience their first period as disablement pensioners af age a. We note that the oldest member to become disabled in the following period are those of age $y = a^R - 1$. We assume that both $r^M_{k,a,t}$ and $r^D_{k,a,t}$ are independent of the age of entry into the pension fund, and we thus have

$$a \leq a^R : N^N_{k,a,e,t} = \left(1 - r^D_{k,a,t}\right)\left(1 - r^M_{k,a,t}\right) N^N_{k,a-1,e,t-1}, \; e < a \tag{6.143}$$

$$a^R < a < a^{LR} : N^N_{k,a,e,t} = \left(1 - r^M_{k,a,t}\right) N^N_{k,a-1,e,t-1}, \; e < a \tag{6.144}$$

This now gives the evolution of the stock of active, non-retired members as

$$y = a^F : N^N_{k,a,t} \equiv N^N_{k,a,a,t} = N^{N,Init}_{k,a,t} \tag{6.145}$$

$$
\begin{aligned}
a^F < a < a^{FR} : N^N_{k,a,t} &\equiv \sum_{e=a^F}^{a} N^N_{k,a,e,t} \\
&= \sum_{e=a^F}^{a-1} N^N_{k,a,e,t} + N^N_{k,a,a,t} \\
&= \sum_{e=a^F}^{a-1} \left(1 - r^D_{k,a,t}\right)\left(1 - r^M_{k,a,t}\right) N^N_{k,a-1,e,t-1} + N^{N,Init}_{k,a,t} \\
&= \left(1 - r^D_{k,a,t}\right)\left(1 - r^M_{k,a,t}\right) \sum_{e=a^F}^{a-1} N^N_{k,a-1,e,t-1} + N^{N,Init}_{k,a,t} \\
&= \left(1 - r^D_{k,a,t}\right)\left(1 - r^M_{k,a,t}\right) N^N_{k,a-1,t-1} + N^{N,Init}_{k,a,t} \qquad (6.146)
\end{aligned}
$$

$$
\begin{aligned}
a^{FR} \leq a \leq a^R : N^N_{k,a,t} \equiv & \sum_{e=a^F}^{a} N^N_{k,a,e,t} \\
= & \sum_{e=a^F}^{a-1} N^N_{k,a,e,t} \\
= & \sum_{e=a^F}^{a-1} \left(1 - r^D_{k,a,t}\right)\left(1 - r^M_{k,a,t}\right) N^N_{k,a-1,e,t-1} \\
= & \left(1 - r^D_{k,a,t}\right)\left(1 - r^M_{k,a,t}\right) \sum_{e=a^F}^{a-1} N^N_{k,a-1,e,t-1} \\
= & \left(1 - r^D_{k,a,t}\right)\left(1 - r^M_{k,a,t}\right) N^N_{k,a-1,t-1} \qquad (6.147)
\end{aligned}
$$

$$
\begin{aligned}
a^R < a < a^{LR} : N^N_{k,a,t} \equiv & \sum_{e=a^F}^{a} N^N_{k,a,e,t} \\
= & \sum_{e=a^F}^{a-1} N^N_{k,a,e,t} \\
= & \sum_{e=a^F}^{a-1} \left(1 - r^M_{k,a,t}\right) N^N_{k,a-1,e,t-1} \\
= & \left(1 - r^M_{k,a,t}\right) \sum_{e=a^F}^{a-1} N^N_{k,a-1,e,t-1} \\
= & \left(1 - r^M_{k,a,t}\right) N^N_{k,a-1,t-1} \qquad (6.148)
\end{aligned}
$$

Active retired members

Non-retired (active) members retire in the age interval between a^{FR} and a^{LR}. Let $N^R_{k,a,r,e,t}$ be the total number of members of category k and age a in period t who retired at age $r \leq a$ and entered at age $e < r$. The total stock of active, non-retired members of category k and age a in period t is therefore

$$N_{k,a,t}^{R} \equiv \sum_{r=a^{FR}}^{a} \sum_{e=a^{F}}^{a^{FR}-1} N_{k,a,r,e,t}^{R} \tag{6.149}$$

where the summation over e stops at age $a^{FR} - 1$ because no new members enter after this age.

For $a > a^{LR}$ we have

$$N_{k,a,t}^{R} = \sum_{r=a^{FR}}^{a-1} \sum_{e=a^{F}}^{a^{FR}-1} N_{k,a,r,e,t}^{R} \tag{6.150}$$

OM UDVIKLING OG OVERGANGSSANDSYNLIGHEDER

We then have the following evolutions

$$a^{FR} \leq a \leq a^{R} : N_{k,a,a,e,t}^{R} = \left(1 - r_{k,a,t}^{D}\right)\left(1 - r_{k,a,t}^{M}\right) r_{k,a,t}^{R} N_{k,a-1,e,t-1}^{N} \tag{6.151}$$

$$a^{R} < y \leq a^{LR} : N_{k,a,a,e,t}^{R} = \left(1 - r_{k,a,t}^{M}\right) r_{k,a,t}^{R} N_{k,a-1,e,t-1}^{N} \tag{6.152}$$

For later use we define

$$\begin{aligned}
a^{FR} &\leq a \leq a^{R} : \\
N_{k,a,t}^{R,Init} &\equiv \sum_{e=a^{F}}^{a^{FR}-1} N_{k,a,a,e,t}^{R} \\
&= \sum_{e=a^{F}}^{a^{FR}-1} \left(1 - r_{k,a,t}^{D}\right)\left(1 - r_{k,a,t}^{M}\right) r_{k,a,t}^{R} N_{k,a-1,e,t-1}^{N} \\
&= \left(1 - r_{k,a,t}^{D}\right)\left(1 - r_{k,a,t}^{M}\right) r_{k,a,t}^{R} \sum_{e=a^{F}}^{a^{FR}-1} N_{k,a-1,e,t-1}^{N} \\
&= \left(1 - r_{k,a,t}^{D}\right)\left(1 - r_{k,a,t}^{M}\right) r_{k,a,t}^{R} N_{k,a-1,t-1}^{N}
\end{aligned} \tag{6.153}$$

$$\begin{aligned}
a^{R} &< a \leq a^{LR} : \\
N_{k,a,t}^{R,Init} &\equiv \sum_{e=a^{F}}^{a^{FR}-1} N_{k,a,a,e,t}^{R} \\
&= \sum_{e=a^{F}}^{a^{FR}-1} \left(1 - r_{k,a,t}^{M}\right) r_{k,a,t}^{R} N_{k,a-1,e,t-1}^{N} \\
&= \left(1 - r_{k,a,t}^{M}\right) r_{k,a,t}^{R} \sum_{e=a^{F}}^{a^{FR}-1} N_{k,a-1,e,t-1}^{N} \\
&= \left(1 - r_{k,a,t}^{M}\right) r_{k,a,t}^{R} N_{k,a-1,t-1}^{N}
\end{aligned} \tag{6.154}$$

And we note that

$$N^{R,Init}_{k,a,t} = \sum_{r=a}^{a} \sum_{e=a^F}^{a^{FR}-1} N^R_{k,a,r,e,t} = \sum_{e=a^F}^{a^{FR}-1} N^R_{k,a,a,e,t} = 0 \text{ for } a > a^{LR} \tag{6.155}$$

The evolution of the number of retired members is given ny

$$N^R_{k,a,r,e,t} = \left(1 - r^M_{a,t}\right) N^R_{k,a-1,r,e,t-1},\ r < a \tag{6.156}$$

For the total number of retired members we have:

$$\begin{aligned}
a &= a^{FR}: \\
N^R_{k,a,t} &\equiv \sum_{r=a^{FR}}^{a} \sum_{e=a^F}^{a^{FR}-1} N^R_{k,a,r,e,t} \\
&= \sum_{r=a}^{a} \sum_{e=a^F}^{a^{FR}-1} N^R_{k,a,r,e,t} \\
&= \sum_{e=a^F}^{a^{FR}-1} N^R_{k,a,a,e,t} \\
&= N^{R,init}_{k,a,t}
\end{aligned} \tag{6.157}$$

$$\begin{aligned}
a^{FR} &< a \leq a^{LR}: \\
N^R_{k,a,t} &\equiv \sum_{r=a^{FR}}^{a} \sum_{e=a^F}^{a^{FR}-1} N^R_{k,a,r,e,t} \\
&= \sum_{r=a^{FR}}^{a-1} \sum_{e=a^F}^{a^{FR}-1} N^R_{k,a,r,e,t} + \sum_{r=a}^{a} \sum_{e=a^F}^{a^{FR}-1} N^R_{k,a,r,e,t} \\
&= \sum_{r=a^{FR}}^{a-1} \sum_{e=a^F}^{a^{FR}-1} \left(1 - r^M_{k,a,t}\right) N^R_{k,a-1,r,e,t-1} + \sum_{e=a^F}^{a^{FR}-1} N^R_{k,a,a,e,t} \\
&= \left(1 - r^M_{k,a,t}\right) \sum_{r=a^{FR}}^{a-1} \sum_{e=a^F}^{a^{FR}-1} N^R_{k,a-1,r,e,t-1} + N^{R,Init}_{k,a,t} \\
&= \left(1 - r^M_{k,a,t}\right) N^R_{k,a-1,t-1} + N^{R,Init}_{k,a,t}
\end{aligned} \tag{6.158}$$

$$\begin{aligned}
a &> a^{LR}: \\
N^{R}_{k,a,t} &\equiv \sum_{r=a^{FR}}^{a} \sum_{e=a^{F}}^{a^{FR}-1} N^{R}_{k,a,r,e,t} \\
&= \sum_{r=a^{FR}}^{a-1} \sum_{e=a^{F}}^{a^{FR}-1} N^{R}_{k,a,r,e,t} \\
&= \sum_{r=a^{FR}}^{a-1} \sum_{e=a^{F}}^{a^{FR}-1} \left(1 - r^{M}_{k,a,t}\right) N^{R}_{k,a-1,r,e,t-1} \\
&= \left(1 - r^{M}_{k,a,t}\right) \sum_{r=a^{FR}}^{a-1} \sum_{e=a^{F}}^{a^{FR}-1} N^{R}_{k,a-1,r,e,t-1} \\
&= \left(1 - r^{M}_{k,a,t}\right) N^{R}_{k,a-1,t-1} \qquad (6.159)
\end{aligned}$$

Disabled members

$$N^{D}_{k,a,t} = \sum_{d=a^{F}+1}^{a} \sum_{e=a^{F}}^{d-1} N^{D}_{k,a,d,e,t} \qquad (6.160)$$

For $a > a^{R}$ we have

$$N^{D}_{k,a,t} = \sum_{d=a^{F}+1}^{R} \sum_{e=a^{F}}^{d-1} N^{D}_{k,a,d,e,t} \qquad (6.161)$$

For later use we define

$$N^{D,Init}_{k,a,t} = \sum_{e=a^{F}}^{a-1} N^{D}_{k,a,a,e,t} \qquad (6.162)$$

We have

$$N^{D}_{k,a,a,e,t} = r^{D}_{k,a,t} \left(1 - r^{M}_{k,a,t}\right) N^{N}_{k,a-1,e,t-1} \qquad (6.163)$$

$$\begin{aligned}
N^{D,Init}_{k,a,t} &\equiv \sum_{e=a^{F}}^{a-1} N^{D}_{k,a,a,e,t} \\
&= \sum_{e=a^{F}}^{a-1} r^{D}_{k,a,t} \left(1 - r^{M}_{k,a,t}\right) N^{N}_{k,a-1,e,t-1} \\
&= r^{D}_{k,a,t} \left(1 - r^{M}_{k,a,t}\right) \sum_{e=a^{F}}^{a-1} N^{N}_{k,a-1,e,t-1} \\
&= r^{D}_{k,a,t} \left(1 - r^{M}_{k,a,t}\right) N^{N}_{k,a-1,t-1} \qquad (6.164)
\end{aligned}$$

The number of disabled members evolves according to

$$N_{k,a,d,e,t} = \left(1 - r^{M}_{k,a,t}\right) N^{D}_{k,a-1,d,e,t-1}, \; a > d \qquad (6.165)$$

And thus we have

$$\begin{aligned} a &= a^F + 1: \\ N^D_{k,a,t} &= \sum_{d=a^F+1}^{a} \sum_{e=a^F}^{d-1} N^D_{k,a,d,e,t} \\ &= \sum_{e=a^F}^{a-1} N^D_{k,a,a,e,t} \\ &= N^{D,Init}_{k,a,t} \end{aligned} \tag{6.166}$$

$$\begin{aligned} a^F + 1 &< a \leq a^R: \\ N^D_{k,a,t} &= \sum_{d=a^F+1}^{a} \sum_{e=a^F}^{d-1} N^D_{k,a,d,e,t} \\ &= \sum_{d=a^F+1}^{a-1} \sum_{e=a^F}^{d-1} N^D_{k,a,d,e,t} + \sum_{e=a^F}^{a-1} N^D_{k,a,d,e,t} \\ &= \sum_{d=a^F+1}^{a-1} \sum_{e=a^F}^{d-1} \left(1 - r^M_{k,a,t}\right) N^D_{k,a-1,d,e,t-1} + N^{D,Init}_{k,a,t} \\ &= \left(1 - r^M_{k,a,t}\right) \sum_{d=a^F+1}^{a-1} \sum_{e=a^F}^{d-1} N^D_{k,a-1,d,e,t-1} + N^{D,Init}_{k,a,t} \\ &= \left(1 - r^M_{k,a,t}\right) N^D_{k,a-1,t-1} + N^{D,Init}_{k,a,t} \end{aligned} \tag{6.167}$$

$$\begin{aligned} a^R &< a \leq a^L: \\ N^D_{k,a,t} &= \sum_{d=a^F+1}^{a} \sum_{e=a^F}^{d-1} N^D_{k,a,d,e,t} \\ &= \sum_{d=a^F+1}^{a-1} \sum_{e=a^F}^{d-1} N^D_{k,a,d,e,t} \\ &= \sum_{d=a^F+1}^{a-1} \sum_{e=a^F}^{d-1} \left(1 - r^M_{k,a,t}\right) N^D_{k,a-1,d,e,t-1} \\ &= \left(1 - r^M_{k,a,t}\right) \sum_{d=a^F+1}^{a-1} \sum_{e=a^F}^{d-1} N^D_{k,a-1,d,e,t-1} \\ &= \left(1 - r^M_{k,a,t}\right) N^D_{k,a-1,t-1} \end{aligned} \tag{6.168}$$

Spouse pensioners

We define

$$N^S_{k,a,t} = \sum_{s=a^F+1}^{a} \sum_{e=a^F}^{s-1} N^{S,N}_{k,a,s,e,t} + \sum_{s=a^F+2}^{a} \sum_{d=a^F+1}^{s-1} \sum_{e=a^F}^{d-1} N^{S,D}_{k,a,s,d,e,t} + \sum_{s=a^{FR}+1}^{a} \sum_{r=a^{FR}}^{s-1} \sum_{e=a^F}^{a^{FR}-1} N^{S,R}_{k,a,s,r,e,t}$$

with adequate corrections depending on actual age!

For later use we define

$$N^{S,N,Init}_{k,a,t} \equiv \sum_{e=a^F}^{a-1} N^{S,N}_{k,a,a,e,t} \tag{6.169}$$

$$N^{S,D,Init}_{k,a,t} \equiv \sum_{d=a^F+1}^{a-1} \sum_{e=a^F}^{d-1} N^{S,D}_{k,a,a,d,e,t} \tag{6.170}$$

$$N^{S,D,Init}_{k,a,t} \equiv \sum_{r=a^{FR}}^{a-1} \sum_{e=a^F}^{a^{FR}-1} N^{S,R}_{k,a,a,r,e,t} \tag{6.171}$$

We have the following evolutions

$$\begin{aligned} N^{S,N}_{k,a,a,e,t} &= r^M_{\hat{k},a,t} N^N_{\hat{k},a-1,e,t-1} \\ N^{S,D}_{k,a,a,e,t} &= r^M_{\hat{k},a,t} N^D_{\hat{k},a-1,d,e,t-1} \\ N^{S,R}_{k,a,a,e,t} &= r^M_{\hat{k},a,t} N^R_{\hat{k},a-1,r,e,t-1} \end{aligned}$$

This gives us

$$\begin{aligned} N^{S,N,Init}_{k,a,t} &\equiv \sum_{e=a^F}^{a-1} N^{S,N}_{k,a,a,e,t} \\ &= \sum_{e=a^F}^{a-1} r^M_{\hat{k},a,t} N^N_{\hat{k},a-1,e,t-1} \\ &= r^M_{\hat{k},a,t} \sum_{e=a^F}^{a-1} N^N_{\hat{k},a-1,e,t-1} \\ &= r^M_{\hat{k},a,t} N^N_{\hat{k},a-1,t-1} \end{aligned}$$

$$\begin{aligned} N^{S,D,Init}_{k,a,t} &\equiv \sum_{d=a^F+1}^{a-1} \sum_{e=a^F}^{d-1} N^{S,D}_{k,a,a,d,e,t} \\ &= \sum_{d=a^F+1}^{a-1} \sum_{e=a^F}^{d-1} r^M_{\hat{k},a,t} N^D_{\hat{k},a-1,d,e,t-1} \\ &= r^M_{\hat{k},a,t} \sum_{d=a^F+1}^{a-1} \sum_{e=a^F}^{d-1} N^D_{\hat{k},a-1,d,e,t-1} \\ &= r^M_{\hat{k},a,t} N^D_{\hat{k},a-1,t-1} \end{aligned}$$

$$\begin{aligned}
N^{S,D,Init}_{k,a,t} &\equiv \sum_{r=a^{FR}}^{a-1} \sum_{e=a^{F}}^{a^{FR}-1} N^{S,R}_{k,a,a,r,e,t} \\
&= \sum_{r=a^{FR}}^{a-1} \sum_{e=a^{F}}^{a^{FR}-1} r^{M}_{\hat{k},a,t} N^{R}_{\hat{k},a-1,r,e,t-1} \\
&= r^{M}_{\hat{k},a,t} \sum_{r=a^{FR}}^{a-1} \sum_{e=a^{F}}^{a^{FR}-1} N^{R}_{\hat{k},a-1,r,e,t-1} \\
&= r^{M}_{\hat{k},a,t} N^{R}_{\hat{k},a-1,t-1}
\end{aligned}$$

for $a > s$ we have

$$N^{S,X}_{k,a,s,x,e,t} = \left(1 - r^{M}_{k,a,t}\right) N^{S,X}_{k,a-1,s,x,e,t-1}$$