Chapter 4

Producer behaviour

The present version of DREAM includes three different production sectors: A private construction sector, an aggregated manufacturing and service sector producing other privately produced goods, and a sector producing government services (henceforth referred to as the government producer).[1]

Producers in the private sectors are called firms. Firms in both private sectors are assumed to produce a good that is an imperfect substitute to goods produced by other firms in the same sector. We assume that a large number of different firms exists, so that the modelling is based on the so-called large group imperfect competition assumption. Production functions are assumed to be identical across firms in a given sector. The individual firms are assumed to take both the wage rate and the length of the working day as given, even though an employers' association negotiates these values with a union of workers, see chapter 5.

The distinction between the manufacturing and the construction sectors implies that the private production in the economy is segmented into one sector producing goods which are traded across borders, and one sector producing goods which are not traded. The fact that goods produced in the construction sector are not traded across borders means that prices in the construction sector are not directly influenced by foreign prices regardless of the elasticities in the foreign trade. Furthermore, the presence of the two production sectors enables a separation of private wealth into residential assets (land and buildings which together form a dwelling), and private financial wealth. This distinction is relevant in the Danish economy, since the Danish capital income tax system separates taxation of financial wealth from taxation of dwellings and land. Imputed rents on dwellings and land receive a favourable tax treatment compared to the tax treatment of returns to financial wealth, and an analysis of the distortions to the savings behavior in the economy therefore necessitates the distinction between the two types of assets.

[1] For simplicity the aggregated manufaturing and service sector is called manufacturing in the following.

An important reason for introducing a government producer in a model that is otherwise highly aggregated with respect to producers, is to obtain a more accurate calculation of the stock of private wealth in the model. The dominant assets of the private households are the households' possession of privately owned buildings and land as well as claims on the value of the capital stock of the private sector. By distinguishing between private and government producers, the considerable value of the government capital stock may be excluded from the assets of households.

In the following we first describe the behavior of firms in the private sectors and solve the problems of profit maximization. We use the subscript j to denote the sector, where j takes the values C for construction and P for manufacturing. We model the large number of firms in both sectors as a continuum of firms and use the subscript h to denote the individual firm, where h belongs to the interval $[0, n_C]$ in the construction sector and $[0, n_P]$ in the manufacturing sector. Unless otherwise stated everything in the following sections 4.1-4.4 holds for both private sectors, $j \in \{C, P\}$, and every firm $h \in [0, n_j]$. In section 4.6 we use the subscript j taking the value G to denote government production. For convenience we shall assume that there is only one government producer.

4.1 The value of the firm

The fundamental behavioural assumption of the firms is that each firm seeks to maximize the value of the outstanding stock of shares. The value of the shares is determined from an arbitrage condition, ensuring that the marginal investor is indifferent between holding bonds and shares. For the model to be able to generate the empirically observed higher pre-tax yield on shares, a risk premium is introduced.[2] The risk premium is modelled as a fixed additional yield to each share. The extra yield is independent of the amount of shares held by the investor in question. Therefore, given the yield difference, shares and bonds are perfect substitutes.

The assumption of perfect mobility of financial capital (and absence of uncertainty) implies that domestic and foreign bonds are perfect substitutes. Therefore a tax-adjusted version of the uncovered interest parity (UIP) holds in the model: Absence of exchange rate movements implies that the domestic interest rate after tax in equilibrium is equal to the foreign interest rate after tax. With a residence-based taxation of personal capital income this implies that the domestic pre-tax interest rate is equal to the foreign pre-tax interest rate.

The Danish capital income taxation is a residence-based tax. Capital income from pension savings is taxed according to a special capital income tax system that implies a favourable

[2] As the model contains no aggregate uncertainty, and therefore, there is no intrinsic motivation for the risk premium in DREAM.

tax treatment compared to other forms of capital income from financial wealth. Therefore a distinction between private investors and institutional investors (i.e. pension funds) is introduced in the model. This implies that three different types of investors may be identified: Foreign investors, domestic private investors and domestic institutional investors.

For a given risk premium the arbitrage condition cannot be fulfilled for all three types of agents simultaneously. As a consequence a given type of investor has to be "appointed" the marginal investor. Given the size and the activity of institutional investors in the Danish market, it is assumed that the behaviour of these investors determine the value of the outstanding stock of shares. This implies that tax rates that apply to pension funds are used to determine the after-tax value in the arbitrage condition of the financial markets. In this way the behaviour of the firm is indirectly affected by the taxation of the institutional investors. Equally important is that by this assumption the capital income taxation of private and foreign investors does not affect the behaviour of firms.

By assumption, foreign investors do not hold any Danish shares. Private investors are assumed to hold a fixed fraction of financial wealth as shares, and the remaining shares are held by the pension funds.

The arbitrage condition for holding shares in firm $h \in [0, n_j]$ of sector $j \in \{C, P\}$ in period τ reads:

$$\left(\left(1 - t_\tau^Z\right) i_\tau + risk_\tau\right) V_{h,j,\tau-1} = \left(1 - t_\tau^Z\right) \left(DIV_{h,j,\tau} + V_{h,j,\tau} - V_{h,j,\tau-1}\right),$$

where i_τ is the nominal interest rate, $risk_\tau$ is the risk premium, t_τ^Z is the tax rate on capital income in the pension fund, $V_{h,j,\tau}$ is the value of outstanding shares in firm h of sector j, and $DIV_{h,j,\tau}$ are dividends paid to shareholders in period τ. Dividing through by $\left(1 - t_\tau^Z\right)$ results in

$$\left(i_\tau + \frac{risk_\tau}{\left(1 - t_\tau^Z\right)}\right) V_{h,j,\tau-1} = DIV_{h,j,\tau} + V_{h,j,\tau} - V_{h,j,\tau-1}, \tag{4.1}$$

The left-hand side of (4.1) is the opportunity cost of holding the value $V_{h,j,\tau-1}$ in shares, i.e. the interest income after tax (including risk premium) that could be gained by holding the value $V_{h,j,\tau-1}$ in bonds. The right-hand side is the income from holding shares, consisting of dividends and a possible capital gain. Note that the formulation implies that the risk premium is independent of the tax rules.

(4.1) is a difference equation in the value of shares, and rearranging the terms yields the following expression for $V_{h,j,\tau-1}$:

$$V_{h,j,\tau-1} = \frac{1}{1 + i_\tau + \frac{risk_\tau}{1-t_\tau^Z}} \left[DIV_{h,j,\tau} + V_{h,j,\tau}\right]. \tag{4.2}$$

Successively leading (4.2) and inserting the result on the right hand side, i.e. repeated forward substitution of V on the right hand side of (4.2), yields

$$V_{h,j,\tau-1} = \sum_{t=\tau}^{\infty} DIV_{h,j,t} \prod_{v=\tau}^{t} \frac{1}{1 + i_v + \frac{risk_v}{1-t_v^Z}}. \tag{4.3}$$

(4.3) shows the market value of the representative firm's shares to be equal to the tax adjusted discounted stream of future dividends. Defining the discount factor

$$R_t \equiv \prod_{v=\tau}^{t} \frac{1}{1 + i_v + \frac{risk_v}{1-t_v^Z}}, \tag{4.4}$$

we may write (4.3) as

$$V_{h,j,\tau-1} = \sum_{t=\tau}^{\infty} R_t DIV_{h,j,t}. \tag{4.5}$$

We incorporate growth in the model by assuming exogenous Harrod-neutral (labour-augmenting) technological progress at the rate g, which therefore constitutes the long run real rate of growth. In addition, inflation in the long run is determined by the exogenous foreign rate of inflation, g^P. Letting $DIV^*_{h,j,t}$ be the dividends of firm h in period t adjusted for growth and inflation between period $\tau - 1$ and period t we have

$$DIV_{h,j,t} = (1+g)^{t-(\tau-1)} \left(1+g^P\right)^{t-(\tau-1)} DIV^*_{h,j,t} \, .$$

Inserting this in equation (4.3) we obtain:

$$V_{h,j,\tau-1} = \sum_{t=\tau}^{\infty} DIV^*_{h,j,t} \prod_{v=\tau}^{t} \frac{(1+g)\left(1+g^P\right)}{1 + i_v + \frac{risk_v}{1-t_v^Z}}.$$

Therefore, a sufficient condition for the value of shares to be finite is that

$$\frac{(1+g)\left(1+g^P\right)}{1 + i_v + \frac{risk_v}{1-t_v^Z}} - 1 < 0 \qquad \forall\, v, \tag{4.6}$$

i.e. that the growth and tax adjusted real yield is positive.[3] This fact is a supplementary reason for the introduction of the risk premium because the growth and tax adjusted real interest rate on bonds has been negative or very close to zero for substantial periods of time.

4.2 The problem of the private firm

We assume that the firm seeks to maximize the value of the outstanding shares, i.e. the firm seeks to maximize the value of the tax-adjusted stream of current and discounted future dividends from the firm. This maximisation is done given the assumption of perfect foresight and subject to the production technology of the firm.

[3] Rearranging (4.6) yields:

$$\frac{1 + i_v + \frac{risk_v}{1-t_v^Z}}{(1+g)\left(1+g^P\right)} - 1 > 0$$

The stream of dividends

Dividends from firm $h \in [0, n_j]$ of sector $j \in \{C, P\}$ are defined as:

$$\begin{aligned} DIV_{h,j,t} = & \left(1 - t_t^{Cor}\right) \\ & \times \Big[P_{h,j,t}^{Y} Y_{h,j,t} + Y_{h,j,t}^{NorthSea} \\ & \quad - P_{h,j,t}^{M} M_{h,j,t} \\ & \quad - \left(1 + t_{j,t}^{Emp} + t_{j,t}^{W} + q_t^{LG}\right) W_t L_{h,j,t}^{D} \\ & \quad - i_t D_{h,j,t-1}^{P} \\ & \quad + s_{j,t}^{EU,P,SetAside} + s_{j,t}^{EU,P,Rural} + o_{j,t}^{G,P,Cap} \\ & \quad - t_{j,t}^{P,Weight} K_{h,j,t-1}^{P,M} - t_{j,t}^{P,Land} P_{j,t-1}^{IPB} K_{h,j,t-1}^{P,B} \Big] \\ & - P_{h,j,t}^{IPM} I_{h,j,t}^{P,M} - P_{h,j,t}^{IPB} I_{h,j,t}^{P,B} \\ & + t_t^{Cor} \delta_{h,j,t}^{P,M,Book} K_{h,j,t-1}^{P,M,Book} + t_t^{Cor} \delta_{h,j,t}^{P,B,Book} K_{h,j,t-1}^{P,B,Book} \\ & + D_{h,j,t}^{P} - D_{h,j,t-1}^{P} \\ & - \left(1 + t_t^{I,I,DutyV} + t_{j,D,t}^{I,I,DutyQ}\right) P_{h,j,t}^{Y} I_{h,j,t}^{P,I} \\ & - \left(1 + t_t^{I,I,DutyV} + t_{j,F,t}^{I,I,DutyQ}\right) \left(1 + t_t^{I,I,Cus}\right) P_{h,j,t}^{F} I_{h,j,t}^{F,I} \\ & - o_{j,t}^{PG,Lump,Cor} - o_{j,t}^{PG,Quasi} - o_{j,t}^{PG,LandRent}, \end{aligned} \tag{4.7}$$

where t_t^{Cor} is the corporate tax rate, $P_{h,j,t}^{Y}$ is the (producer) price, and $Y_{h,j,t}$ is net production (net of capital installation costs). $Y_{h,j,t}^{NorthSea}$ is a special income stream representing pure profits from the extraction and sales of gas and oil reserves from the North Sea, cf. below. $P_{h,j,t}^{M}$ is the price (index) of materials, $M_{h,j,t}$ is the input of materials, $t_{j,t}^{Emp}$ is the rate of labour market contributions paid by employers, $t_{j,t}^{W}$ is the wage sum tax rate, q_t^{LG} are contributions paid by employers to the LG fund (Lønmodtagernes Garantifond), W_t is the wage paid by producers per unit of labour employed, $L_{h,j,t}^{D}$ is the amount of labour demanded by the firm, i_t is the nominal interest rate, $D_{h,j,t-1}^{P}$ is the stock of corporate debt at the beginning of period t, $s_{j,t}^{EU,P,SetAside}$ is a lump sum subsidy for set aside schemes (braklægningsstøtte) financed by the EU, $s_{j,t}^{EU,P,Rural}$ is a lump sum subsidy to rural land (hektarstøtte) financed by the EU[4], $t_{j,t}^{P,Weight}$ is the tax rate on the use of motor vehicles (vægtafgift), which is modelled as a tax on the stock of machinery capital, $K_{h,j,t-1}^{P,M}$, because motor vehicles as such are not a production factor in DREAM,[5] $t_{j,t}^{P,Land}$ is the tax rate applied to taxes on land value (grundskyld), which

[4]Both $s_{j,t}^{EU,P,SetAside}$ and $s_{j,t}^{EU,P,Rural}$ are relevant only for the agricultural sector, which in DREAM is a part of the manufacturing sector. The same is true for $Y_{h,j,t}^{NorthSea}$. For symmetry reasons, however, we invoke these terms in the expressions for dividends in both private production sectors, but $s_{j,t}^{EU,P,SetAside}$, $s_{j,t}^{EU,P,Rural}$ and $Y_{h,j,t}^{NorthSea}$ are set equal to zero in the construction sector, i.e. for $j \in \{C\}$.

[5]The superscript P on $K_{h,i,t-1}^{P,M}$ is used to distinguish capital used in production from capital used elsewhere. In the present version of DREAM, however, machinery is used only in production, while buildings are used

is modelled as a tax on the value of the building capital stock, $P^{IPB}_{h,j,t-1}K^{P,B}_{h,j,t-1}$, because land is not a factor of production, $P^{IPB}_{h,j,t-1}$ is the price (index) of building capital and $K^{P,B}_{h,j,t-1}$ is the building capital stock, $P^{IPM}_{h,j,t}$ is the price (index) of machinery investments, $I^{P,M}_{h,j,t}$ are machinery investments, $P^{IPB}_{h,j,t}$ is the price (index) of building investments, $I^{P,B}_{h,j,t}$ are building investments, and thus the terms $P^{IPM}_{h,j,t}I^{P,M}_{h,j,t}$ and $P^{IPB}_{h,j,t}I^{P,B}_{h,j,t}$ denote expenses to machinery and building investments. $\delta^{P,M,Book}_{h,j,t}$ is the rate of depreciation on machinery capital allowed by the tax system, $K^{P,M,Book}_{h,j,t-1}$ is the book value (defined according to the tax system) of machinery capital, and $\delta^{P,B,Book}_{h,j,t}$ and $K^{P,B,Book}_{h,j,t-1}$ are the corresponding terms for building capital. Thus, the terms $t^{Cor}_{t}\delta^{P,M,Book}_{h,j,t}K^{P,M,Book}_{h,j,t-1}$ and $t^{Cor}_{t}\delta^{P,B,Book}_{h,j,t}K^{P,B,Book}_{h,j,t-1}$ denote the tax values of depreciation on machinery and building capital. The expression $\left(D^{P}_{h,j,t} - D^{P}_{h,j,t-1}\right)$ denotes the change in debt levels during the period. The next two lines represent costs of inventory and livestock investments (from domestic and foreign suppliers), which are included only in the calibration year to replicate the values from the National Accounts. Since these terms are exogenous and not chosen by the firm, they are of no importance for the solution to the firm's problem. The same can be said for the three terms in the last line which represent lump-sum transfers from the producers to the government sector. The first term is a calibration term ensuring that the total corporate tax revenues of the calibration year correspond to the National Accounts. The last two terms represent National Account payments of land rent and withdrawals from quasi corporations, cf. chapter 7.

Income from energy extraction in the North Sea

Part of present Danish aggregate production consists of extraction and sales of gas and oil reserves in the North Sea. Extraction of these resources are expected to decline considerably during the next decades, affecting e.g. government revenues. In general, relative declines of specific production sectors need not affect production and revenue in a model like DREAM with homogeneous aggregate production sectors. It is assumed that production factors are alternatively used in other connections. However, in the case of the extraction of nonrenewable resources, the resource itself is an independent production factor which generates a specific return (the pure resource rent) besides the returns to capital, labour and materials, and consequently the decline in future energy extraction will presumably generate a smaller resource rent. To capture this effect in the model, it is chosen to model the effect of energy extraction simply as an exogenous income stream $Y^{NorthSea}_{h,j,t}$ to firms in the manufacturing sector which is earned independently of the use of factor inputs. The income stream thus represents the pure resource rent and is calculated on the basis of assumptions made by the Danish tax and energy authorities.

both in production and in consumption.

Financing the activites of the firm

It is assumed that the firm finances all activity by borrowing and by retaining earnings, which may profitably be invested within the firm, and pays out only the remainder as dividends. Dividends are therefore determined residually, and firms abstain from financing increases in activity by issuing new shares. To avoid corner solutions in the optimal financing decision of the firm (which appears due to the non-neutrality of the tax system together with the absence of uncertainty in the model), we model corporate debt as a fixed fraction, $w_{j,t}^{DP}$, of the replacement value of the capital stock:

$$D_{h,j,t}^{P} = w_{j,t}^{DP} \left(P_{h,j,t}^{IPM} K_{h,j,t}^{PM} + P_{h,j,t}^{IPB} K_{h,j,t}^{PB} \right). \tag{4.8}$$

Observe that this exogenous financing rule of the firm implies that dividends may temporarily become negative in case of a sufficiently large boost in investments. In this case the firm collects negative dividends from shareholders.[6]

4.2.1 Production function and nest structure of the firm

Production in every firm in both private sectors (and also government production) is assumed to take the same functional form, but the parameters of the production function may vary between sectors. Specifically production is given by a (nested) CES production function, and figure 1 shows the nest structure.[7]

Gross production of firm h in sector j, $Y_{h,j,t}^{Gross}$, is given by a two-factor CES production function using materials, $M_{h,j,t}$, and value-added, $H_{h,j,t}$ ('Gross' means that installation costs have not been subtracted). The actual production function has the following form:

$$Y_{h,j,t}^{Gross} = \left(\left(\mu_{h,j}^{M}\right)^{\frac{1}{\sigma_{h,j}^{Y}}} \left(M_{h,j,t}\right)^{\frac{\sigma_{h,j}^{Y}-1}{\sigma_{h,j}^{Y}}} + \left(\mu_{h,j}^{H}\right)^{\frac{1}{\sigma_{h,j}^{Y}}} \left(H_{h,j,t}\right)^{\frac{\sigma_{h,j}^{Y}-1}{\sigma_{h,j}^{Y}}} \right)^{\frac{\sigma_{h,j}^{Y}}{\sigma_{h,j}^{Y}-1}},$$

where $\sigma_{h,j}^{Y}$ is the elasticity of substitution between materials and the composite value-added input, and $\mu_{h,j}^{M}$ and $\mu_{h,j}^{H}$ are calibrated distribution parameters.

Materials $M_{h,j,t}$ are produced by combining (via a three-factor CES production function) materials originating from construction ($M_{h,j,C,t}^{1}$), manufacturing ($M_{h,j,P,t}^{1}$) and production of government services ($M_{h,j,G,t}^{1}$) This and subsequent production functions are shown in appendix .... In the following nest materials from manufacturing, $M_{h,j,P,t}^{1}$, are produced by combining (using a two-factor CES production function) domestic and foreign materials, $M_{h,j,P,D,t}^{2}$ and $M_{h,j,P,F,t}^{2}$

[6] Given the calibration of the model this phenomenon is unlikely to appear for shocks of "normal" size.

[7] Indices denoting producing firm, industry and time, i.e. h, j and t, are not shown in figure 1.

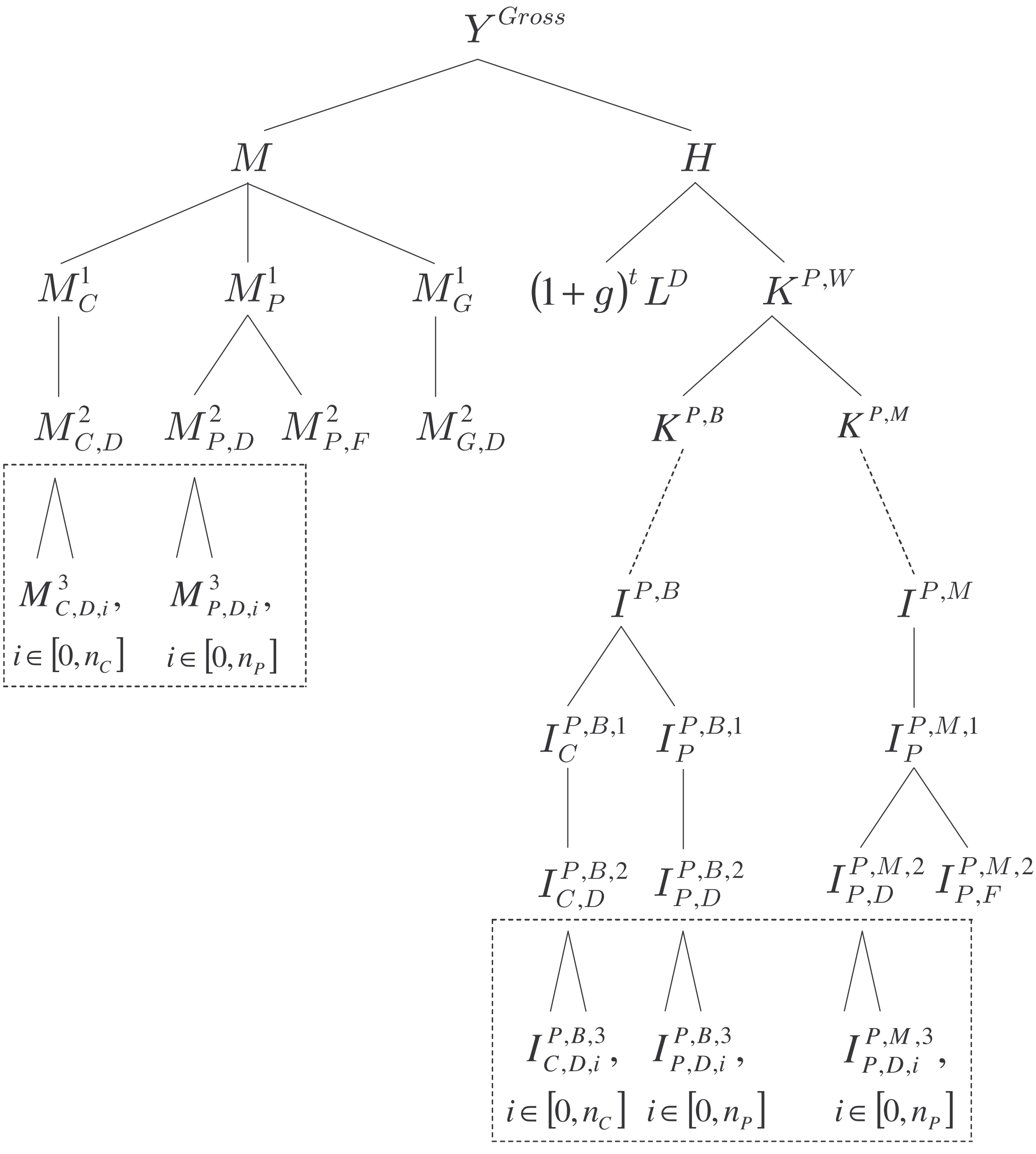

Figure 4.1: Nest Structure

respectively, while materials from the construction sector and production of government services are domestically produced only, $M^1_{h,j,C,t} = M^2_{h,j,C,D,t}$ and $M^1_{h,j,G,t} = M^2_{h,j,G,D,t}$. Finally, domestically produced materials from the construction and manufacturing sectors, $M^2_{h,j,C,D,t}$ and $M^2_{h,j,P,D,t}$, are CES-indices of materials bought from every firm, i, in the construction, $M^3_{C,D,i}$, and manufacturing, $M^3_{P,D,i}$, sectors respectively.

Value added, $H_{h,j,t}$, is a two-factor CES-index of labour and capital, $K^{P,W}_{h,j,t}$. Because we assume Harrod-neutral technological progress at the rate g, labour input is given by $(1+g)^t L^D_{h,j,t}$.

4.2.2 Investment and capital stock

Capital, $K^{P,W}_{h,j,t}$, is produced using a two-factor CES production function with building and machinery capital, $K^{P,B}_{h,j,t}$ and $K^{P,M}_{h,j,t}$, as inputs. Note that the capital stock is ultimo-dated in the model, and consequently $K^{P,B}_{h,j,t-1}$ enters into the production function in period t, etc.

$K^{P,M}_{h,j,t}$ and $K^{P,B}_{h,j,t}$ are subject to exponential decay and evolve according to the accumulation equations:

$$K^{P,M}_{h,j,t} = \left(1-\delta^{P,M}_{h,j,t}\right) K^{P,M}_{h,j,t-1} + I^{P,M}_{h,j,t}, \quad (4.9)$$

$$K^{P,B}_{h,j,t} = \left(1-\delta^{P,B}_{h,j,t}\right) K^{P,B}_{h,j,t-1} + I^{P,B}_{h,j,t}, \quad (4.10)$$

where $\delta^{P,M}_{h,j,t}$ and $\delta^{P,B}_{h,j,t}$ are depreciation rates for machinery and building capital respectively.

The composition of machinery and building investments is also shown in Figure 1.1. Building investment, $I^{P,B}_{h,j,t}$, is given as a two-factor CES index of capital goods produced in the construction and manufacturing sectors, $I^{P,B,1}_{h,j,C,t}$ and $I^{P,B,1}_{h,j,P,t}$, and the following nest shows that these both derive from domestic production only, i.e. $I^{P,B,1}_{h,j,C,t} = I^{P,B,2}_{h,j,C,D,t}$ and $I^{P,B,1}_{h,j,P,t} = I^{P,B,2}_{h,j,P,D,t}$. The final nest shows that domestically produced building capital goods originating from the construction and manufacturing sectors, $I^{P,B,2}_{h,j,C,D,t}$ and $I^{P,B,2}_{h,j,P,D,t}$, are CES-indices of goods bought from every firm, i, in the construction and manufacturing sectors respectively.

Machinery investments, $I^{P,M}_{h,j,t}$, all originate from manufacturing, $I^{P,M}_{h,j,t} = I^{P,M,1}_{h,j,P,t}$, and $I^{P,M,1}_{h,j,P,t}$ is a two-factor CES index of domestic and foreign goods, $I^{P,M,2}_{h,j,P,D,t}$ and $I^{P,M,2}_{h,j,P,F,t}$. Finally, domestically produced machinery capital goods, $I^{P,M,2}_{h,j,P,D,t}$, is a CES index of goods bought from every firm, i, in the manufacturing sector.

It should be noted that the bottom nests of materials, building and machinery investments, i.e. inputs bought from individual domestic firms of the construction and manufacturing sectors, are not invoked explicitly in DREAM, i.e. the variables do not appear in the model, which is illustrated in Figure 1.1 by the dotted boxes. The bottom nests are important in the formal derivation, however, because they show how (part of) the demand facing each firm is derived and how each firm is one among many engaged in large group competition.[8]

Installation costs

The installation of both building and machinery capital is costly and the cost depends positively on the amount of investments relative to the existing capital stock. The installation cost functions are given as

$$IC^{M}_{h,j}\left(I^{PM}_{h,j,t},K^{PM}_{h,j,t-1}\right) = k^{I,P,M}_{h,j}\frac{\left(I^{PM}_{h,j,t}\right)^2}{K^{PM}_{h,j,t-1}}, \tag{4.11}$$

$$IC^{B}_{h,j}\left(I^{PB}_{h,j,t},K^{PB}_{h,j,t-1}\right) = k^{I,P,B}_{h,j}\frac{\left(I^{PB}_{h,j,t}\right)^2}{K^{PB}_{h,j,t-1}}, \tag{4.12}$$

where $k^{I,P,M}_{h,j}$ and $k^{I,P,B}_{h,j}$ are scale parameters. Installation costs are thus increasing and strictly convex in the level of investment and decreasing in the level of existing capital stock. The fact that installation costs are increasing and strictly convex in the level of investments means that it is optimal for the firm, i.e. it lowers costs, to spread desired investments over time rather than to make instantaneous adjustments of the capital stock. In addition, both $IC^{M}_{h,j}\left(I^{PM}_{h,j,t},K^{PM}_{h,j,t-1}\right)$ and $IC^{B}_{h,j}\left(I^{PB}_{h,j,t},K^{PB}_{h,j,t-1}\right)$ are homogeneous of degree one in order to ensure that the model can generate a steady state. The assumption of first degree homogeneity implies that e.g. a doubling of investments and the capital stock will double installation costs.

Subtracting installation costs from gross production yields net output:

$$Y_{h,j,t} = Y^{Gross}_{h,j,t} - k^{I,P,M}_{h,j}\frac{\left(I^{PM}_{h,j,t}\right)^2}{K^{PM}_{h,j,t-1}} - k^{I,P,B}_{h,j}\frac{\left(I^{PB}_{h,j,t}\right)^2}{K^{PB}_{h,j,t-1}}. \tag{4.13}$$

The book value of capital evolves according to:

$$K^{P,M,Book}_{h,j,t} = \left(1-\delta^{P,M,Book}_{h,j,t}\right)K^{P,M,Book}_{h,j,t-1} + P^{IPM}_{h,j,t}I^{P,M}_{h,j,t}, \tag{4.14}$$

$$K^{P,B,Book}_{h,j,t} = \left(1-\delta^{P,B,Book}_{h,j,t}\right)K^{P,B,Book}_{h,j,t-1} + P^{IPB}_{h,j,t}I^{P,B}_{h,j,t}, \tag{4.15}$$

where $\delta^{P,M,Book}_{h,j,t}$ and $\delta^{P,B,Book}_{h,j,t}$ are the depreciation rates allowed by the tax laws.

4.2.3 Demand faced by the firm

The demand facing the representative firm, h, in sector j is given by

$$Y_{h,j,t} = \varrho^{E_j}_{h,j}\left(\frac{P^{Y}_{h,j,t}}{\bar{P}^{Y}_{j,t}}\right)^{-E_j}\Theta_{j,t}\,, \tag{4.16}$$

[8] Demand for goods produced by individual firms also comes from households, the government sector and the foreign sector.

where

$$\bar{P}^{Y}_{j,t} = \left(\int_0^{n_j} \varrho^{E_j}_{h,j} \left(P^{Y}_{h,j,t} \right)^{1-E_j} dh \right)^{\frac{1}{1-E_j}}. \tag{4.17}$$

Total demand for the products of each firm is composed of demand by other firms (for intermediate inputs), by consumers, by the government and by the foreign sector. $\Theta_{j,t}$ represents the value of total demand for the output of sector j, and $\varrho^{E_j}_{h,j}$ is a scale parameter. Demand is negatively correlated with the price $P^{Y}_{h,j,t}$, but the price elasticity $-E_j$ is finite so that the demand curve is downward-sloping. Consequently, E_j is also an expression for the market power of the firm.

From (4.16) we obtain

$$P^{Y}_{h,j,t} = \varrho_{h,j} \Theta^{\frac{1}{E_j}}_{j,t} \bar{P}^{Y}_{j,t} Y^{-\frac{1}{E_j}}_{h,j,t},$$

and using this expression, total revenue in firm h is given by

$$TR_{h,j,t} \left(Y_{h,j,t} \right) \equiv P^{Y}_{h,j,t} Y_{h,j,t} = \varrho_{h,j} \Theta^{\frac{1}{E_j}}_{j,t} \bar{P}^{Y}_{j,t} Y^{\frac{E_j-1}{E_j}}_{h,j,t}. \tag{4.18}$$

Because each firm is one among many, it disregards the effect of its own price on the price index of the sector. Therefore marginal revenue is given as

$$\begin{aligned} \frac{\partial TR_{h,j,t} \left(Y_{h,j,t} \right)}{\partial Y_{h,j,t}} &= \frac{E_j - 1}{E_j} \varrho_{h,j} \Theta^{\frac{1}{E_j}}_{j,t} \bar{P}^{Y}_{j,t} Y^{-\frac{1}{E_j}}_{h,j,t} \\ &= \frac{E_j - 1}{E_j} P^{Y}_{h,j,t}, \end{aligned}$$

which is just the well-known relationship between marginal revenue and price.

For later use it turns out to be convenient to think of marginal revenue as an "optimization price", $P^{O}_{h,j,t}$, and we therefore have

$$P^{O}_{h,j,t} \equiv \frac{\partial TR_{h,j,t} \left(Y_{h,j,t} \right)}{\partial Y_{h,j,t}} \tag{4.19}$$

$$= \frac{E_j - 1}{E_j} P^{Y}_{h,j,t}. \tag{4.20}$$

4.3 Solving the problem of the private firms

In the arbitrary period τ the firm strives to maximize the value of the outstanding shares, $V_{h,j,\tau-1}$, subject to the constraints facing the firm, i.e. (4.7) – (4.16).[9] The problem can be simplified slightly using the following manipulations: Disregarding the intermediate CES

[9] $V_{h,j,\tau-1} = \sum_{t=\tau}^{\infty} R_t DIV_{h,j,t}$ is measured in present value in period $\tau - 1$ and it therefore may seem more natural in period τ to maximise the value of $\left(1 + i_\tau + \frac{risk_\tau}{1-t^Z_\tau} \right) V_{h,j,\tau-1}$. However, this leads to the same result as maximisation of $V_{h,j,\tau-1}$, since the firm takes $1 + i_\tau + \frac{risk_\tau}{1-t^Z_\tau}$ to be independent of its actions.

production functions for value-added, $H_{h,j,t}$, and total weighted capital stock, $K^{P,W}_{h,j,t}$, gross production is a function of materials, labour, and the stocks of machinery and building capital:

$$Y^{Gross}_{h,j,t} = F\left[K^{P,M}_{h,j,t-1}, K^{P,B}_{h,j,t-1}, M_{h,j,t}, (1+g)^t L^D_{h,j,t}\right]. \tag{4.21}$$

From eq. (4.8) using eq. (4.9) and (4.10) we have:

$$D^P_{h,j,t} = w^{DP}_{j,t}\left(P^{IPM}_{h,j,t}\left[\left(1-\delta^{PM}_{h,j,t}\right)K^{PM}_{h,j,t-1} + I^{PM}_{h,j,t}\right] + P^{IPB}_{h,j,t}\left[\left(1-\delta^{PB}_{h,j,t}\right)K^{PB}_{h,j,t-1} + I^{PB}_{h,j,t}\right]\right) \tag{4.22}$$

Inserting (4.18), then (4.13) and afterwards (4.21) into the definition of dividends of eq. (4.7) and further inserting equation (4.22) (instead of $D^P_{h,j,t}$) and equation (4.8) lagged one period (instead of $D^P_{h,j,t-1}$; this is done twice) into the result, we get

$$\begin{aligned}
DIV_{j,t} = {} & \left(1-t^{Cor}_t\right) \qquad (4.23)\\
& \times TR_{h,j,t}\left(F\left[K^{PM}_{h,j,t-1}, K^{P,B}_{h,j,t-1}, M_{h,j,t}, (1+g)^t L^D_{h,j,t}\right] - \frac{k^{I,P,M}_{h,j}\left(I^{PM}_{h,j,t}\right)^2}{K^{PM}_{h,j,t-1}} - \frac{k^{I,P,B}_{h,j}\left(I^{PB}_{h,j,t}\right)^2}{K^{PB}_{h,j,t-1}}\right)\\
& + \left(1-t^{Cor}_t\right) Y^{NorthSea}_{h,j,t}\\
& - \left(1-t^{Cor}_t\right) P^M_{h,j,t} M_{h,j,t}\\
& - \left(1-t^{Cor}_t\right)\left(1+t^{Emp}_{j,t}+t^W_{j,t}+q^{LG}_t\right) W_t L^D_{h,j,t}\\
& - \left(1-t^{Cor}_t\right) i_t \cdot w^{DP}_{j,t}\left(P^{IPM}_{h,j,t-1}K^{PM}_{h,j,t-1} + P^{IPB}_{h,j,t-1}K^{PB}_{h,j,t-1}\right)\\
& + \left(1-t^{Cor}_t\right)\left(s^{EU,P,SetAside}_{j,t} + s^{EU,P,Rural}_{j,t} + o^{G,P,Cap}_{j,t}\right)\\
& - \left(1-t^{Cor}_t\right) t^{P,Weight}_{j,t} K^{P,M}_{h,j,t-1} - \left(1-t^{Cor}_t\right) t^{P,Land}_{j,t} P^{IPB}_{h,j,t-1} K^{P,B}_{h,j,t-1}\\
& - P^{IPM}_{h,j,t} I^{P,M}_{h,j,t} - P^{IPB}_{h,j,t} I^{P,B}_{h,j,t}\\
& + t^{Cor}_t \delta^{P,M,Book}_{h,j,t} K^{P,M,Book}_{h,j,t-1} + t^{Cor}_t \delta^{P,B,Book}_{h,j,t} K^{P,B,Book}_{h,j,t-1}\\
& + w^{DP}_{j,t}\left(P^{IPM}_{h,j,t}\left[\left(1-\delta^{PM}_{h,j,t}\right)K^{PM}_{h,j,t-1} + I^{PM}_{h,j,t}\right] + P^{IPB}_{h,j,t}\left[\left(1-\delta^{PM}_{h,j,t}\right)K^{PB}_{h,j,t-1} + I^{PB}_{h,j,t}\right]\right)\\
& \quad - w^{DP}_{j,t-1}\left(P^{IPM}_{h,j,t-1}K^{PM}_{h,j,t-1} + P^{IPB}_{h,j,t-1}K^{PB}_{h,j,t-1}\right)\\
& - \left(1+t^{I,I,DutyV}_t + t^{I,I,DutyQ}_{j,D,t}\right) P^Y_{h,j,t} I^{P,I}_{h,j,t}\\
& - \left(1+t^{I,I,DutyV}_t + t^{I,I,DutyQ}_{j,F,t}\right)\left(1+t^{I,I,Cus}_t\right) P^F_{h,j,t} I^{F,I}_{h,j,t}\\
& - o^{PG,Lump,Cor}_{j,t} - o^{PG;Quasi}_{j,t} - o^{PG,LandRent}_{j,t}.
\end{aligned}$$

The profit maximization problem may be solved as a two stage problem where the firm in the first stage chooses the optimal values of the indices $M_{h,j,t}$, $I^{P,M}_{h,j,t}$ and $I^{P,B}_{h,j,t}$ and the input of labour, $L^D_{h,j,t}$ (intertemporal optimization). At the second stage the firm chooses the optimal compositions of the indices $M_{h,j,t}$, $I^{P,M}_{h,j,t}$ and $I^{P,B}_{h,j,t}$ (intratemporal optimization). These optimization problems are solved in the following.

4.3.1 Intertemporal optimization

Formally the profit maximization problem of the individual firm may now be stated as:

$$\max_{\left(M_{h,j,t},L^{D}_{h,j,t},I^{P,M}_{h,j,t},I^{P,B}_{h,j,t}\right)^{\infty}_{t=\tau}} V_{h,j,\tau-1}=\sum_{t=\tau}^{\infty}R_t DIV_{h,j,t}$$

$$\text{s.t. } (4.9)\,,\ (4.10)\,,\ (4.14)\,,\ (4.15)\,,\ (4.23)\,.$$

The Hamiltonian associated with this problem is:

$$\begin{aligned}
\mathcal{H} \;=\; & R_t\cdot\Big\{\ \left(1-t^{Cor}_t\right)\\
&\times TR_{h,j,t}\left(F\left[K^{PM}_{h,j,t-1},K^{P,B}_{h,j,t-1},M_{h,j,t},(1+g)^t L^D_{h,j,t}\right]-\frac{k^{I,P,M}_{h,j}\left(I^{PM}_{h,j,t}\right)^2}{K^{PM}_{h,j,t-1}}-\frac{k^{I,P,B}_{h,j}\left(I^{PB}_{h,j,t}\right)^2}{K^{PB}_{h,j,t-1}}\right)\\
&+\left(1-t^{Cor}_t\right)Y^{NorthSea}_{h,j,t}-\left(1-t^{Cor}_t\right)P^{M}_{h,j,t}M_{h,j,t}\\
&-\left(1-t^{Cor}_t\right)\left(1+t^{Emp}_{j,t}+t^{W}_{j,t}+q^{LG}_t\right)W_tL^D_{h,j,t}\\
&-\left(1-t^{Cor}_t\right)i_t\cdot w^{DP}_{j,t}\left(P^{IPM}_{h,j,t-1}K^{PM}_{h,j,t-1}+P^{IPB}_{h,j,t-1}K^{PB}_{h,j,t-1}\right)\\
&+\left(1-t^{Cor}_t\right)\left(s^{EU,P,SetAside}_{j,t}+s^{EU,P,Rural}_{j,t}+o^{G,P,Cap}_{j,t}\right)\\
&-\left(1-t^{Cor}_t\right)t^{P,Weight}_{j,t}K^{P,M}_{h,j,t-1}-\left(1-t^{Cor}_t\right)t^{P,Land}_{j,t}P^{IPB}_{h,j,t-1}K^{P,B}_{h,j,t-1}\\
&-P^{IPM}_{h,j,t}I^{P,M}_{h,j,t}-P^{IPB}_{h,j,t}I^{P,B}_{h,j,t}\\
&+t^{Cor}_t\delta^{P,M,Book}_{h,j,t}K^{P,M,Book}_{h,j,t-1}+t^{Cor}_t\delta^{P,B,Book}_{h,j,t}K^{P,B,Book}_{h,j,t-1}\\
&+w^{DP}_{j,t}\left(P^{IPM}_{h,j,t}\left[\left(1-\delta^{PM}_{h,j,t}\right)K^{PM}_{h,j,t-1}+I^{PM}_{h,j,t}\right]+P^{IPB}_{h,j,t}\left[\left(1-\delta^{PM}_{h,j,t}\right)K^{PB}_{h,j,t-1}+I^{PB}_{h,j,t}\right]\right)\\
&-w^{DP}_{j,t-1}\left(P^{IPM}_{h,j,t-1}K^{PM}_{h,j,t-1}+P^{IPB}_{h,j,t-1}K^{PB}_{h,j,t-1}\right)\\
&-\left(1+t^{I,I,DutyV}_t+t^{I,I,DutyQ}_{j,D,t}\right)P^{Y}_{h,j,t}I^{P,I}_{h,j,t}\\
&-\left(1+t^{I,I,DutyV}_t+t^{I,I,DutyQ}_{j,F,t}\right)\left(1+t^{I,I,Cus}_t\right)P^{F}_{h,j,t}I^{F,I}_{h,j,t}\\
&-o^{PG,Lump,Cor}_{j,t}-o^{PG,Quasi}_{j,t}-o^{PG,LandRent}_{j,t}\ \Big\}\\
&+R_t\cdot Q^{KPM}_{h,j,t}\left[\left(1-\delta^{PM}_{h,j,t}\right)K^{PM}_{h,j,t-1}+I^{PM}_{h,j,t}\right]\\
&+R_t\cdot Q^{KPB}_{h,j,t}\left[\left(1-\delta^{PB}_{h,j,t}\right)K^{PB}_{h,j,t-1}+I^{PB}_{h,j,t}\right]\\
&+R_t\cdot Q^{KPM,Book}_{h,j,t}\left[\left(1-\delta^{PM,Book}_{h,j,t}\right)K^{PM,Book}_{h,j,t-1}+P^{IPM}_{h,j,t}I^{PM}_{h,j,t}\right]\\
&+R_t\cdot Q^{KPB,Book}_{h,j,t}\left[\left(1-\delta^{PB,Book}_{h,j,t}\right)K^{PB,Book}_{h,j,t-1}+P^{IPB}_{h,j,t}I^{PB}_{h,j,t}\right],
\end{aligned}$$

where $M_{h,j,t}$, $L^D_{h,j,t}$, $I^{PM}_{h,j,t}$ and $I^{PB}_{h,j,t}$ are control variables, $K^{PM}_{h,j,t-1}$, $K^{PB}_{h,j,t-1}$, $K^{PM,Book}_{h,j,t-1}$ and $K^{PB,Book}_{h,j,t-1}$ are state variables and $Q^{KPM}_{h,j,t}$, $Q^{KPB}_{h,j,t}$, $Q^{KPM,Book}_{h,j,t}$ and $Q^{KPB,Book}_{h,j,t}$ are the shadow prices (co-state variables) associated with the state variables in the order just mentioned. The first order conditions for maximisation are (using the definition $P^O_{h,j,t} \equiv \frac{\partial TR_{h,j,t}\left(Y_{h,j,t}\right)}{\partial Y_{h,j,t}}$ of (4.19)):

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial M_{h,j,t}} &= R_t \cdot \left(1 - t^{Cor}_t\right)\left[P^O_{h,j,t}\frac{\partial F}{\partial M_{h,j,t}} - P^M_{h,j,t}\right] \\ &= 0, \qquad (4.24)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial L^D_{h,j,t}} &= R_t \cdot \left(1 - t^{Cor}_t\right)\left[P^O_{h,j,t}\frac{\partial F}{\partial\left((1+g)^t L^D_{h,j,t}\right)}(1+g)^t - \left(1 + t^{Emp}_{j,t} + t^W_{j,t} + q^{LG}_t\right)W_t\right] \\ &= 0, \qquad (4.25)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial I^{PM}_{h,j,t}} &= R_t \cdot \left[w^{DP}_{j,t}P^{IPM}_{h,j,t} - \left(1 - t^{Cor}_t\right)P^O_{h,j,t}2k^{I,P,M}_{h,j}\frac{I^{PM}_{h,j,t}}{K^{PM}_{h,j,t-1}} - P^{IPM}_{h,j,t}\right] \\ &\quad + R_t \cdot Q^{KPM}_{h,j,t} + R_t \cdot Q^{KPM,Book}_{h,j,t}P^{IPM}_{h,j,t} \\ &= 0, \qquad (4.26)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial I^{PB}_{h,j,t}} &= R_t \cdot \left[w^{DP}_{j,t}P^{IPB}_{h,j,t} - \left(1 - t^{Cor}_t\right)P^O_{h,j,t}2k^{I,P,B}_{h,j}\frac{I^{PB}_{h,j,t}}{K^{PB}_{h,j,t-1}} - P^{IPB}_{h,j,t}\right] \\ &\quad + R_t \cdot Q^{KPB}_{h,j,t} + R_t \cdot Q^{KPB,Book}_{h,j,t}P^{IPB}_{h,j,t} \\ &= 0, \qquad (4.27)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial K^{PM}_{h,j,t-1}} &= R_t \cdot \Bigg[\left(1 - t^{Cor}_t\right)P^O_{h,j,t}\left(\frac{\partial F}{\partial K^{PM}_{h,j,t-1}} + k^{I,P,M}_{h,j}\left(\frac{I^{PM}_{h,j,t}}{K^{PM}_{h,j,t-1}}\right)^2\right) \\ &\qquad - \left(1 - t^{Cor}_t\right)i_t w^{DP}_{j,t}P^{IPM}_{h,j,t-1} \\ &\qquad - \left(1 - t^{Cor}_t\right)t^{P,Weight}_{j,t} \\ &\qquad + w^{DP}_{j,t}P^{IPM}_{h,j,t}\left(1 - \delta^{PM}_{h,j,t}\right) \\ &\qquad - w^{DP}_{j,t-1}P^{IPM}_{h,j,t-1} \Bigg] \\ &\quad + R_t \cdot Q^{KPM}_{h,j,t}\left(1 - \delta^{PM}_{h,j,t}\right) \\ &= R_{t-1}Q^{KPM}_{h,j,t-1}, \qquad (4.28)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial K^{PB}_{h,j,t-1}} &= R_t \cdot \Bigg[\left(1 - t^{Cor}_t\right)P^O_{h,j,t}\left(\frac{\partial F}{\partial K^{PB}_{h,j,t-1}} + k^{I,P,B}_{h,j}\left(\frac{I^{PB}_{h,j,t}}{K^{PB}_{h,j,t-1}}\right)^2\right) \\ &\qquad - \left(1 - t^{Cor}_t\right)i_t w^{DP}_{j,t}P^{IPB}_{h,j,t-1} \\ &\qquad - \left(1 - t^{Cor}_t\right)t^{P,Land}_{j,t}P^{IPB}_{h,j,t-1} \\ &\qquad + w^{DP}_{j,t}P^{IPB}_{h,j,t}\left(1 - \delta^{PM}_{h,j,t}\right) \\ &\qquad - w^{DP}_{j,t-1}P^{IPB}_{h,j,t-1} \Bigg] \\ &\quad + R_t \cdot Q^{KPB}_{h,j,t}\left(1 - \delta^{PB}_{h,j,t}\right) \\ &= R_{t-1}Q^{KPB}_{h,j,t-1}, \qquad (4.29)\end{aligned}$$

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial K^{PM,Book}_{h,j,t-1}} &= R_t\cdot t^{Cor}_t\delta^{PM,Book}_{h,j,t}+R_t\cdot Q^{KPM,Book}_{h,j,t}\left(1-\delta^{PM,Book}_{h,j,t}\right)\\ &= R_{t-1}Q^{KPM,Book}_{h,j,t-1},\end{aligned}\tag{4.30}$$

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial K^{PB,Book}_{h,j,t-1}} &= R_t t^{Cor}_t\delta^{PB,Book}_{h,j,t}+R_t Q^{KPB,Book}_{h,j,t}\left(1-\delta^{PB,Book}_{h,j,t}\right)\\ &= R_{t-1}Q^{KPB,Book}_{h,j,t-1}.\end{aligned}\tag{4.31}$$

We now simplify the set of equations (4.24) – (4.31) , making use of the fact that

$$\frac{R_{t-1}}{R_t}=1+i_t+\frac{risk_t}{1-t^Z_t},$$

(the definition of R_t is given in eq. (4.4)). Furthermore, we define

$$\begin{aligned}P^O_{h,j,t}\frac{\partial F}{\partial K^{PM}_{h,j,t-1}} &\equiv P^{KPM}_{h,j,t},\\ P^O_{h,j,t}\frac{\partial F}{\partial K^{PB}_{h,j,t-1}} &\equiv P^{KPB}_{h,j,t},\end{aligned}$$

i.e. $P^{KPM}_{h,j,t}$ and $P^{KPB}_{h,j,t}$ are the marginal revenue products of machinery and building capital, respectively. We then have:

$$P^O_{h,j,t}\frac{\partial F}{\partial K^{PM}_{h,j,t-1}}=P^{KPM}_{h,j,t},\tag{4.32}$$

$$P^O_{h,j,t}\frac{\partial F}{\partial K^{PB}_{h,j,t-1}}=P^{KPB}_{h,j,t},\tag{4.33}$$

while eqs. (4.24) – (4.31) become

$$P^O_{h,j,t}\frac{\partial F}{\partial M_{h,j,t}}=P^M_{h,j,t},\tag{4.34}$$

$$P^O_{h,j,t}\frac{\partial F}{\partial\left((1+g)^t L^D_{h,j,t}\right)}=\left(1+t^{Emp}_{j,t}+t^W_{j,t}+q^{LG}_t\right)\frac{W_t}{(1+g)^t},\tag{4.35}$$

$$\left(1-t^{Cor}_t\right)2k^{I,P,M}_{h,j}\frac{I^{PM}_{h,j,t}}{K^{PM}_{h,j,t-1}}\frac{P^O_{h,j,t}}{P^{IPM}_{h,j,t}}+1-w^{DP}_{j,t}=\frac{Q^{KPM}_{h,j,t}}{P^{IPM}_{h,j,t}}+Q^{KPM,Book}_{h,j,t},\tag{4.36}$$

$$\left(1-t^{Cor}_t\right)2k^{I,P,B}_{h,j}\frac{I^{PB}_{h,j,t}}{K^{PB}_{h,j,t-1}}\frac{P^O_{h,j,t}}{P^{IPB}_{h,j,t}}+1-w^{DP}_{j,t}=\frac{Q^{KPB}_{h,j,t}}{P^{IPB}_{h,j,t}}+Q^{KPB,Book}_{h,j,t},\tag{4.37}$$

$$\begin{aligned}&\Bigg\{\left(1-t^{Cor}_t\right)\left[P^{KPM}_{h,j,t}+k^{I,P,M}_{h,j}\left(\frac{I^{PM}_{h,j,t}}{K^{PM}_{h,j,t-1}}\right)^2P^O_{h,j,t}-i_t w^{DP}_{j,t-1}P^{IPM}_{h,j,t-1}-t^{P,Weight}_{j,t}\right]\\ &\quad+w^{DP}_{j,t}P^{IPM}_{h,j,t}\left(1-\delta^{PM}_{h,j,t}-\frac{P^{IPM}_{h,j,t-1}}{P^{IPM}_{h,j,t}}\right)\Bigg\}\\ &=\left(i_t+\frac{risk_t}{1-t^Z_t}\right)Q^{KPM}_{h,j,t-1}+\delta^{PM}_{h,j,t}Q^{KPM}_{h,j,t}-\left(Q^{KPM}_{h,j,t}-Q^{KPM}_{h,j,t-1}\right),\end{aligned}\tag{4.38}$$

$$\left\{(1-t_t^{Cor})\left[P_{h,j,t}^{KPB}+k_{h,j}^{I,P,B}\left(\frac{I_{h,j,t}^{PB}}{K_{h,j,t-1}^{PB}}\right)^2 P_{h,j,t}^{O}-\left(i_t w_{j,t-1}^{DP}+t_{j,t}^{P,Land}\right)P_{h,j,t-1}^{IPB}\right]\right.$$

$$\left.+w_{j,t}^{DP}P_{h,j,t}^{IPB}\left(1-\delta_{h,j,t}^{PB}-\frac{P_{h,j,t-1}^{IPB}}{P_{h,j,t}^{IPB}}\right)\right\} \quad (4.39)$$

$$= \left(i_t+\frac{risk_t}{1-t_t^Z}\right)Q_{h,j,t-1}^{KPB}+\delta_{h,j,t}^{PB}Q_{h,j,t}^{KPB}-\left(Q_{h,j,t}^{KPB}-Q_{h,j,t-1}^{KPB}\right),$$

$$t_t^{Cor}\delta_{h,j,t}^{PM,Book} \quad (4.40)$$

$$= \left(i_t+\frac{risk_t}{1-t_t^Z}\right)Q_{h,j,t-1}^{KPM,Book}+\delta_{h,j,t}^{PM,Book}Q_{h,j,t}^{KPM,Book}-\left(Q_{h,j,t}^{KPM,Book}-Q_{h,j,t-1}^{KPM,Book}\right),$$

$$t_t^{Cor}\delta_{h,j,t}^{PB,Book} \quad (4.41)$$

$$= \left(i_t+\frac{risk_t}{1-t_t^Z}\right)Q_{h,j,t-1}^{KPB,Book}+\delta_{h,j,t}^{PB,Book}Q_{h,j,t}^{KPB,Book}-\left(Q_{h,j,t}^{KPB,Book}-Q_{h,j,t-1}^{KPB,Book}\right).$$

(4.34) and (4.35) are standard static first order conditions. (4.34) states that in optimum the marginal revenue product of materials is equal to the price of materials.

Likewise, eq. (4.35) implicitly gives the demand for labour by requiring that the marginal revenue product of labour is equal to the marginal cost of labour. Observe that in a stationary situation, where $M_{k,j,t}$, $K_{h,j,t}^{PM}$, $K_{h,j,t}^{PB}$ and $L_{h,j,t}^{D}$ are constant, the wage rate is increased by a factor $(1+g)$ each period. It is a standard property of Harrod-neutral technological progress that the rate of technological progress is fully shifted onto the wage costs.

The equations governing the optimal investment decisions are given by equations (4.36) and (4.37), which have the same form and the same interpretation. Focusing on eq. (4.36) for machinery investments, the interpretation is the usual requirement that in optimum the marginal cost of investment should be equal to the marginal benefit of investment. The marginal cost is given on the left hand side and consists of the direct and indirect costs of investments. The direct cost to the owners of the firm is given by $1-w_{j,t}^{DP}$, where $w_{j,t}^{DP}$ is the debt ratio of the firm and therefore denotes the fraction of investment which is financed by borrowing, and therefore does not constitute a direct cost to the owners. The indirect cost is equal to the (after corporate tax) value of the cost of installation, $2k_{h,j}^{I,P,M}\frac{I_{h,j,t}^{PM}}{K_{h,j,t-1}^{PM}}\frac{P_{h,j,t}^{O}}{P_{h,j,t}^{IPM}}$. The benefits of investments are given on the right hand side and are two-fold. First, investment adds to the capital stock of the firm. The shadow price $Q_{h,j,t}^{KPM}$ is the marginal value of an additional unit of capital to the owners of the firm. Second, investments, and therefore increases in the capital stock, adds to the depreciation allowance of the firm. The shadow price $Q_{h,j,t}^{KPM,Book}$ measures the marginal value of a unit depreciation allowance to the owners of the firm. The total benefits are the sum of the two.

The determination of shadow prices are given by the last four equations, which are the first order conditions for the stock variables of the firm. Starting backwards, eq. (4.40) and (4.41) are of the same form and therefore have the same interpretation. Focusing on eq. (4.40), we solve this for $Q^{KPM,Book}_{h,j,t-1}$ and lead the resulting expression one period. This yields:

$$Q^{KPM,Book}_{h,j,t} = \frac{t^{Cor}_{t+1}\delta^{PM,Book}_{h,j,t+1} + \left(1-\delta^{PM,Book}_{h,j,t+1}\right) Q^{KPM,Book}_{h,j,t+1}}{1+i_{t+1}+\frac{risk_{t+1}}{1-t^{Z}_{t+1}}}.$$

Solving this difference equation forward yields:

$$Q^{KPM,Book}_{h,j,t} = \frac{t^{Cor}_{t+1}\delta^{PM,Book}_{h,j,t+1}}{1+i_{t+1}+\frac{risk_{t+1}}{1-t^{Z}_{t+1}}} + \sum_{v=2}^{\infty} t^{Cor}_{t+v}\delta^{PM,Book}_{h,j,t+v} \frac{\prod_{s=1}^{v-1}\left(1-\delta^{PM,Book}_{h,j,t+s}\right)}{\prod_{s=1}^{v}\left(1+i_{t+s}+\frac{risk_{t+1}}{1-t^{Z}_{t+s}}\right)}.$$

Thus $Q^{KPM,Book}_{h,j,t}$ is given as the tax-adjusted, discounted future stream of the value of depreciation allowance of one unit of machinery capital in period t. The discounting includes both the depreciation of the book value and the tax-adjusted interest rate. $Q^{KPM,Book}_{h,j,t}$ therefore measures the increase in the market value of the firm's shares due to the depreciation allowance associated with the marginal unit of capital, which is kept in the firm forever.

Turning to eq. (4.38) and (4.39), these are of a similar form and interpretation, and we shall focus on eq. (4.38), which may be solved for $Q^{KPM}_{h,j,t-1}$.[10] Leading the resulting expression one period yields:

$$Q^{KPM}_{h,j,t} = \frac{\Lambda_{h,j,t+1} + \left(1-\delta^{PM}_{h,j,t+1}\right) Q^{KPM}_{h,j,t+1}}{\left(1+i_t+\frac{risk_{t+1}}{1-t^{Z}_{t+1}}\right)},$$

where

$$\begin{aligned}\Lambda_{h,j,t} \\ \equiv \Bigg\{ & \left(1-t^{Cor}_t\right)\left[P^{KPM}_{h,j,t} + k^{I,P,M}_{h,j}\left(\frac{I^{PM}_{h,j,t}}{K^{PM}_{h,j,t-1}}\right)^2 P^{O}_{h,j,t} - i_t w^{DP}_{j,t-1}P^{IPM}_{h,j,t-1} - t^{P,Weight}_{j,t}\right] \\ & + w^{DP}_{j,t}P^{IPM}_{h,j,t}\left(\left(1-\delta^{PM}_{h,j,t}\right) - \frac{P^{IPM}_{h,j,t-1}}{P^{IPM}_{h,j,t}}\right)\Bigg\}\end{aligned}$$

is the tax-adjusted increase in dividends in period t associated with a marginal increase in the machinery capital stock $K^{PM}_{h,j,t-1}$. The effect on dividends has several sources. First, there is a direct positive effect from marginal revenue, where $P^{KPM}_{h,j,t}$ is the marginal revenue product of the capital stock. Second, there is a positive effect from the cost of installation, which is decreased as the capital stock increases. Next there are two negative effects coming from the exogenous financing decision, where the increased capital stock increases debt and therefore interest payments by $i_t w^{DP}_{j,t-1}P^{IPM}_{h,j,t-1}$, and from taxation at the rate $t^{P,Weight}_{j,t}$ of the increased

[10]The only qualitative difference between eqs. (4.38) and (4.39) is due to the fact that the taxation of the use of motor vehicles is a tax on the *stock* of (machinery) capital, whereas the tax base of property taxes is the *value* of the (building) capital stock. Therefore, $t^{P,Land}_{j,t}$ in equation (4.39) is multiplied by the price $P^{IPB}_{h,j,t-1}$ whereas $t^{P,Weight}_{j,t}$ in eq. (4.38) is not multiplied by a price.

capital stock. Finally, the increased capital stock increases the change in the *value* of the capital stock and thus the change in the firm's debt, which is a fixed fraction of the value of the capital stock. Changes in the debt affects dividends, and this is captured by the last term.

Solving the difference equation for $Q^{KPM}_{h,j,t}$ forward yields:

$$Q^{KPM}_{h,j,t} = \frac{1}{\left(1+i_t+\frac{risk_{t+1}}{1-t^Z_{t+1}}\right)}\Lambda_{h,j,t+1} + \sum_{v=2}^{\infty} \frac{\prod_{s=1}^{v-1}\left(1-\delta^{PM}_{h,j,t+s}\right)}{\prod_{s=1}^{v}\left(1+i_{t+s-1}+\frac{risk_{t+s}}{1-t^Z_{t+s}}\right)}\Lambda_{h,j,t+v}.$$

$Q^{KPM}_{h,j,t}$ is thus given as the discounted future stream of marginal increases in dividends, where the discounting contains both physical depreciation and the risk-adjusted rate of interest.

An alternative formulation of the maximization problem using CES solution methods

A closer inspection of equations (4.32) – (4.35) reveals that these are identical to the first order conditions for the following problem

$$\begin{aligned} &\max_{K^{PM}_{h,j,t-1},K^{PB}_{h,j,t-1},M_{h,j,t},L^D_{h,j,t}} \Pi_{h,j,t} \\ \text{s.t.}\quad &\Pi_{h,j,t} = P^O_{h,j,t}Y^{Gross}_{h,j,t} - P^{KPM}_{h,j,t}K^{PM}_{h,j,t-1} - P^{KPB}_{h,j,t}K^{PB}_{h,j,t-1} - P^M_{h,j,t}M_{h,j,t} - W_t L^D_{h,j,t} \\ &Y^{Gross}_{h,j,t} = F\left[K^{PM}_{h,j,t-1}, K^{PB}_{h,j,t-1}, M_{h,j,t}, (1+g)^t L^D_{h,j,t}\right], \end{aligned}$$

and given the production nest structure we know the solution to be

$$M_{h,j,t} = \mu^M_{h,j}\left(\frac{P^M_{h,j,t}}{P^O_{h,j,t}}\right)^{-\sigma^Y_{h,j}} Y^{Gross}_{h,j,t}, \tag{4.42}$$

$$H_{h,j,t} = \mu^H_{h,j}\left(\frac{P^H_{h,j,t}}{P^O_{h,j,t}}\right)^{-\sigma^Y_{h,j}} Y^{Gross}_{h,j,t}, \tag{4.43}$$

$$(1+g)^t L^D_{h,j,t} = \mu^L_{h,j}\left(\frac{\left(1+t^{Emp}_{j,t}+t^W_{j,t}\right)\frac{W_t}{(1+g)^t}}{P^H_{h,j,t}}\right)^{-\sigma^H_{h,j}} H_{h,j,t}, \tag{4.44}$$

$$K^{P,W}_{h,j,t} = \mu^{KP}_{h,j}\left(\frac{P^{KPW}_{h,j,t}}{P^H_{h,j,t}}\right)^{-\sigma^H_{h,j}} H_{h,j,t}, \tag{4.45}$$

$$K^{PM}_{h,j,t-1} = \mu^{KPM}_{h,j}\left(\frac{P^{KPM}_{h,j,t}}{P^{KPW}_{h,j,t}}\right)^{-\sigma^K_{h,j}} K^{P,W}_{h,j,t}, \tag{4.46}$$

$$K^{PB}_{h,j,t-1} = \mu^{KPB}_{h,j}\left(\frac{P^{KPB}_{h,j,t}}{P^{KP,W}_{h,j,t}}\right)^{-\sigma^K_{h,j}} K^{P,W}_{h,j,t}, \tag{4.47}$$

where

$$P^{O}_{h,j,t} = \left(\mu^{M}_{h,j}\left(P^{M}_{h,j,t}\right)^{1-\sigma^{Y}_{h,j}} + \mu^{H}_{h,j}\left(P^{H}_{h,j,t}\right)^{1-\sigma^{Y}_{h,j}}\right)^{\frac{1}{1-\sigma^{Y}_{h,j}}}, \quad (4.48)$$

$$P^{H}_{h,j,t} = \left(\mu^{L}_{h,j}\left(\left(1+t^{Emp}_{j,t}+t^{W}_{j,t}\right)\frac{W_t}{(1+g)^t}\right)^{1-\sigma^{H}_{h,j}} + \mu^{KP}_{h,j}\left(P^{KP,W}_{h,j,t}\right)^{1-\sigma^{H}_{h,j}}\right)^{\frac{1}{1-\sigma^{H}_{h,j}}}, \quad (4.49)$$

$$P^{KP,W}_{h,j,t} = \left(\mu^{KPM}_{h,j}\left(P^{KPM}_{h,j,t}\right)^{1-\sigma^{K}_{h,j}} + \mu^{KPB}_{h,j}\left(P^{KPB}_{h,j,t}\right)^{1-\sigma^{K}_{h,j}}\right)^{\frac{1}{1-\sigma^{K}_{h,j}}}. \quad (4.50)$$

For a given gross production, (4.42) and (4.43) determine the demand for the indices of materials and value added. Given these, (4.44) determines labour demand, and the three next equations together with (4.50) then yield the values of $K^{P,W}_{h,j,t}$ and the three prices $P^{KPW}_{h,j,t}$, $P^{KPM}_{h,j,t}$ and $P^{KPB}_{h,j,t}$.

What remains are equations (4.36)−(4.41), which yield the values of $I^{PM}_{h,j,t}$, $I^{PB}_{h,j,t}$, $Q^{KPB}_{h,j,t}$, $Q^{KPM}_{h,j,t}$, $Q^{KPM,Book}_{h,j,t}$ and $Q^{KPB,Book}_{h,j,t}$.

Finally we need the relation between the marginal revenue, or the optimization price, $P^{O}_{h,j,t}$ and the chosen product price of the firm, which according to eq. (4.20) is given by:

$$P^{O}_{h,j,t} = \frac{E_j - 1}{E_j} P^{Y}_{h,j,t}. \quad (4.51)$$

4.3.2 Intratemporal optimization

Given the values of the indices of materials, machinery and building investments resulting from the intertemporal optimization described above, each firm composes these indices using standard CES cost-minimizing methods. The resulting equations are shown in this section for all three nests; the derivation method can be found in appendix ?.

Use of materials

First nest Materials, $M_{h,j,t}$, are produced using inputs from the construction, manufacturing and government sectors. Inputs are chosen to minimize costs associated with obtaining the index $M_{h,j,t}$:

$$\min_{\left(M^{1}_{h,j,k,t}\right)} \sum_{k\in\{C,P,G\}} P^{M1}_{h,j,k,t} M^{1}_{h,j,k,t}$$

$$\text{s.t.} \qquad M_{h,j,t} = \left(\sum_{k\in\{C,P,G\}} \left(\mu^{M1}_{h,j,k}\right)^{\frac{1}{\sigma^{M}_{h,j}}} \left(M^{1}_{h,j,k,t}\right)^{\frac{\sigma^{M}_{h,j}-1}{\sigma^{M}_{h,j}}}\right)^{\frac{\sigma^{M}_{h,j}}{\sigma^{M}_{h,j}-1}},$$

where $M^1_{h,j,k,t}$ is materials originating from sector k and used in firm h of sector j, and $P^{M1}_{h,j,k,t}$ is the price (index) of $M^1_{h,j,k,t}$. The solution to the cost minimization problem is given by

$$M^1_{h,j,k,t} = \mu^{M1}_{h,j,k} \left(\frac{P^{M1}_{h,j,k,t}}{P^M_{h,j,t}} \right)^{-\sigma^M_{h,j}} M_{h,j,t}, \quad k \in \{C, P, G\}, \tag{4.52}$$

$$P^M_{h,j,t} = \left(\sum_{k \in \{C,P,G\}} \mu^{M1}_{h,j,k} \left(P^{M1}_{h,j,k,t} \right)^{1-\sigma^M_{h,j}} \right)^{\frac{1}{1-\sigma^M_{h,j}}}. \tag{4.53}$$

Second nest Only materials from manufacturing, $M^1_{h,j,P,t}$, are produced using both domestic and foreign materials, but formally we let both $M^1_{h,j,C,t}$, $M^1_{h,j,P,t}$ and $M^1_{h,j,G,t}$ be indices of domestic and foreign materials and minimize the costs associated with obtaining these indices:

$$\min_{\left(M^2_{h,j,k,c,t}\right)} \sum_{c \in \{D,F\}} P^{M2}_{h,j,k,c,t} M^2_{h,j,k,c,t} \quad , k \in \{C, P, G\}$$

$$\text{s.t.} \qquad M^1_{h,j,k,t} = \left(\sum_{c \in \{D,F\}} \left(\mu^{M2}_{h,j,k,c} \right)^{\frac{1}{\sigma^{M1}_{h,j}}} \left(M^2_{h,j,k,c,t} \right)^{\frac{\sigma^{M1}_{h,j}-1}{\sigma^{M1}_{h,j}}} \right)^{\frac{\sigma^{M1}_{h,j}}{\sigma^{M1}_{h,j}-1}},$$

where $M^2_{h,j,k,c,t}$ are materials originating from domestic ($c = D$) or foreign ($c = F$) sector k. The solution is

$$M^2_{h,j,k,c,t} = \mu^{M2}_{h,j,k,c} \left(\frac{P^{M2}_{h,j,k,c,t}}{P^{M1}_{h,j,k,t}} \right)^{-\sigma^{M1}_{h,j}} M^1_{h,j,k,t}, \quad c \in \{D, F\}, \quad k \in \{C, P, G\}, \tag{4.54}$$

$$P^{M1}_{h,j,k,t} = \left(\sum_{c \in \{D,F\}} \mu^{M2}_{h,j,k,c} \left(P^{M2}_{h,j,k,c,t} \right)^{1-\sigma^{M1}_{h,j}} \right)^{\frac{1}{1-\sigma^{M1}_{h,j}}}, \quad k \in \{C, P, G\}. \tag{4.55}$$

By setting

$$\mu^{M2}_{h,j,C,F} = \mu^{M2}_{h,j,G,F} = 0 \text{ and } \mu^{M2}_{h,j,C,D} = \mu^{M2}_{h,j,G,D} = 1,$$

as it is done in the actual computer program implementing DREAM, equations (4.54) and (4.55) yield

$$\begin{aligned} M^2_{h,j,C,F,t} &= M^2_{h,j,G,F,t} = 0, \\ M^2_{h,j,C,D,t} &= M^1_{h,j,C,t}, \\ M^2_{h,j,G,D,t} &= M^1_{h,j,G,t}, \\ P^{M1}_{h,j,C,t} &= P^{M2}_{h,j,C,D,t}, \\ P^{M1}_{h,j,G,t} &= P^{M2}_{h,j,G,D,t}. \end{aligned}$$

I.e. materials from construction and government production are of domestic origin only.

The price paid for imported material inputs is equal to the foreign price, P_t^F, including customs taxes, $t_{j,t}^{M,Cus}$, which may depend on the industry in which the imported goods are used as inputs, but does not depend on the specific firm using the goods. We therefore have:[11]

$$\begin{aligned} P_{h,j,k,F,t}^{M2} &= \left(1 + t_{j,t}^{Res} - s_{j,t}^{G,P,Dwe} - s_{j,t}^{G,P,Res} - s_{j,t}^{EU,P,Res} + t_{j,t}^{M,VAT} + t_{j,t}^{M,DutyV}\right. \\ &\quad \left. + t_{j,D,k,t}^{M,DutyQ} - s_{j,t}^{GP,Spe} - s_{j,t}^{EU,P,Spe}\right)\left(1 + t_{j,t}^{M,Cus}\right) P_t^F, \\ k &\in \{C,P,G\},\ j \in \{C,P,G\}. \end{aligned} \tag{4.56}$$

We maintain the use of subscript h in $P_{h,j,k,F,t}^{M2}$ to stress the fact that it is the price paid by firm h, but examination of the right hand side of equation (4.56) reveals the fact that $P_{h,j,k,F,t}^{M2}$ is the same for all firms, h, of industry j.

Third nest For materials produced domestically by firms in the construction and manufacturing sectors, $M_{h,j,k,D,t}^2$, is an aggregate of materials produced by all firms in the delivering sector (including the demanding firm if it belongs to the delivering sector, i.e. firms may use their own products as part of inputs). For $k \in \{C, P\}$ each firm in each sector thus solves

$$\min_{\left(M_{h,j,k,D,i,t}^3\right)_{i\in[0,n_k]}} \int_0^{n_k} P_{h,j,k,D,i,t}^{M3} M_{h,j,k,D,i,t}^3 di$$

$$\text{s.t.} \qquad M_{h,j,k,D,t}^2 = \left(\int_0^{n_k} \left(\varrho_{i,k}\right)^{\frac{1}{E_k}} \left(M_{h,j,k,D,i,t}^3\right)^{\frac{E_k-1}{E_k}} di\right)^{\frac{E_k}{E_k-1}},$$

where $M_{h,j,k,D,i,t}^3$ are materials originating from domestic firm i of sector k, and $P_{h,j,k,D,i,t}^{M3}$ is the price paid. The solution is given by

$$M_{h,j,k,D,i,t}^3 = \varrho_{i,k} \left(\frac{P_{h,j,k,D,i,t}^{M3}}{P_{h,j,k,D,t}^{M2}}\right)^{-E_k} M_{h,j,k,D,t}^2, \quad i \in [0, n_k], \quad k \in \{C, P\}, \tag{4.57}$$

$$P_{h,j,k,D,t}^{M2} = \left(\int_0^{n_k} \varrho_{i,k} \left(P_{h,j,k,D,i,t}^{M3}\right)^{1-E_k} di\right)^{\frac{1}{1-E_k}}, \quad k \in \{C, P\}. \tag{4.58}$$

The price paid by firm h for goods bought from firm i is equal to the producer price of firm i including taxes, $t_{j,k,t}^{IPM}$, which may depend on both the industry using the goods and the industry producing the goods, but not on individual firms. We therefore have:

$$\begin{aligned} P_{h,j,k,D,i,t}^{M3} &= \left(1 + t_{j,k,t}^{Res} - s_{j,t}^{G,P,Dwe} - s_{j,t}^{G,P,Res} - s_{j,t}^{EU,P,Res} + t_{j,t}^{M,VAT} + t_{j,t}^{M,DutyV}\right. \\ &\quad \left. + t_{j,D,k,t}^{M,DutyQ} - s_{j,t}^{GP,Spe} - s_{j,t}^{EU,P,Spe}\right) P_{i,k,t}^Y, \\ i &\in [0, n_k], \quad k \in \{C, P\}. \end{aligned} \tag{4.59}$$

Again, we maintain the use of subscript h in $P_{h,j,k,D,i,t}^{M3}$ to emphasize that it is the price paid by firm h, though $P_{h,j,k,D,i,t}^{M3}$ is in fact equal for all firms of industry j.

[11] Only goods from manufacturing are imported, i.e. only $P_{h,j,P,F,t}^{M2}$ is needed in the model, but again we define $P_{h,j,k,F,t}^{M2}$ for all $k \in \{C, P, G\}$ in order to mimic the equations used in the actual computer code.

Machinery investments

The method for deriving the optimal composition and price (indices) of machinery investments is exactly equal to that of materials. Unlike materials, machinery investments all originate from the (domestic or foreign) manufacturing sector, but again formally the equations used comprise all sectors and the distribution parameters are used as dummies to exclude the non-relevant sectors. Consequently, each firm solves the following problem:

$$\min_{\left(I^{PM1}_{h,j,k,t}\right)} \sum_{k\in\{P\}} P^{IPM1}_{h,j,k,t} I^{PM1}_{h,j,k,t}$$

$$\text{s.t.} \qquad I^{PM}_{h,j,t} = \left(\sum_{k\in\{P\}} \left(\mu^{IPM1}_{h,j,k}\right)^{\frac{1}{\sigma^{IPM}_{h,j}}} \left(I^{PM1}_{h,j,k,t}\right)^{\frac{\sigma^{IPM}_{h,j}-1}{\sigma^{IPM}_{h,j}}} \right)^{\frac{\sigma^{IPM}_{h,j}}{\sigma^{IPM}_{h,j}-1}},$$

with the solution

$$I^{PM1}_{h,j,k,t} = \mu^{IPM1}_{h,j,k} \left(\frac{P^{IPM1}_{h,j,k,t}}{P^{IPM}_{h,j,t}} \right)^{-\sigma^{IPM}_{h,j}} I^{PM}_{h,j,t}, \quad k \in \{P\}, \tag{4.60}$$

$$P^{IPM}_{h,j,t} = \left(\sum_{k\in\{P\}} \mu^{IPM1}_{h,j,k} \left(P^{IPM1}_{h,j,k,t}\right)^{1-\sigma^{IPM}_{h,j}} \right)^{\frac{1}{1-\sigma^{IPM}_{h,j}}}, \tag{4.61}$$

where $\mu^{IPM1}_{h,j,P} = 1$.[12]

In the second nest, the solution correspondingly becomes

$$I^{PM2}_{h,j,k,c,t} = \mu^{IPM2}_{h,j,k,c} \left(\frac{P^{IPM2}_{h,j,k,c,t}}{P^{IPM1}_{h,j,k,t}} \right)^{-\sigma^{IPM1}_{h,j}} I^{PM1}_{h,j,k,t}, \quad c \in \{D,F\}, \quad k \in \{P\}, \tag{4.62}$$

$$P^{IPM1}_{h,j,k,t} = \left(\sum_{c\in\{D,F\}} \mu^{IPM2}_{h,j,k,c} \left(P^{IPM2}_{h,j,k,c,t}\right)^{1-\sigma^{IPM1}_{h,j}} \right)^{\frac{1}{1-\sigma^{IPM1}_{h,j}}}, \quad k \in \{P\}. \tag{4.63}$$

Again, the price paid for imported machinery investment goods is equal to the foreign price, P^F_t, including taxes, $t^{IM,Cus}_{j,t}$:

$$\begin{aligned} P^{IPM2}_{h,j,k,F,t} &= \left(1 + t^{P,Reg}_{j,k,t} + t^{IM,VAT}_{j,t} + t^{IM,DutyV}_{j,t} + t^{IM,DutyQ}_{j,D,k,t} \right. \\ &\quad \left. - s^{G,IM,Spe}_{j,t} - s^{EU,IM,Spe}_{j,t}\right) \left(1 + t^{IM,Cus}_{j,t}\right) P^F_t, \\ k &\in \{P\}. \end{aligned} \tag{4.64}$$

[12]It thus all amounts to:

$$\begin{aligned} I^{PM1}_{h,j,P,t} &= I^{PM}_{h,j,t} \\ P^{IPM}_{h,j,t} &= P^{IPM1}_{h,j,P,t} \end{aligned}$$

In the third nest, the solution becomes

$$I^{PM3}_{h,j,k,D,i,t} = \varrho_{i,k} \left(\frac{P^{IPM3}_{h,j,k,D,i,t}}{P^{IPM2}_{h,j,k,D,t}} \right)^{-E_k} I^{PM2}_{h,j,k,D,t}, \quad i \in [1, n_k], \quad k \in \{P\}, \tag{4.65}$$

$$P^{IPM2}_{h,j,k,D,t} = \left(\int_0^{n_k} \varrho_{i,k} \left(P^{IPM3}_{h,j,k,D,i,t} \right)^{1-E_k} di \right)^{\frac{1}{1-E_k}}, \quad k \in \{P\}. \tag{4.66}$$

$$\begin{aligned} P^{IPM3}_{h,j,k,D,i,t} &= \Big(1 + t^{P,Reg}_{j,k,t} + t^{IM,VAT}_{j,t} + t^{IM,DutyV}_{j,t} + t^{IM,DutyQ}_{j,D,k,t} \\ &\quad - s^{G,IM,Spe}_{j,t} - s^{EU,IM,Spe}_{j,t} \Big) P^{Y}_{i,k,t}, \\ i &\in [0, n_k], \quad k \in \{P\}. \end{aligned} \tag{4.67}$$

The price paid by firm h for goods bought from firm i is equal to the producer price of firm i including taxes, $t^{IPM}_{j,k,t}$, which may depend on both the industry using the goods and the industry producing the goods, but not on individual firms. We maintain the use of subscript h in $P^{IPM3}_{h,j,k,D,i,t}$ to stress the fact that it is the price paid by firm h, but examination of the right hand side of eq. (4.67) reveals the fact that $P^{IPM3}_{h,j,k,D,i,t}$ is the same for all firms, h, of industry j.

Building investments

The demand for building investments is parallel to machinery investments, except that building investments originate from both the construction and the manufacturing sectors, but are purchased only domestically. Again, these asymmetries are handled by the distribution parameters acting as dummy variables. The solution to the optimization problem becomes:

$$I^{PB1}_{h,j,k,t} = \mu^{IPB1}_{h,j,k} \left(\frac{P^{IPB1}_{h,j,k,t}}{P^{IPB}_{h,j,t}} \right)^{-\sigma^{IPB}_{h,j}} I^{PB}_{h,j,t}, \quad k \in \{C, P\}, \tag{4.68}$$

$$P^{IPB}_{h,j,t} = \left(\sum_{k \in \{C,P\}} \mu^{IPB1}_{h,j,k} \left(P^{IPB1}_{h,j,k,t} \right)^{1-\sigma^{IPB}_{h,j}} \right)^{\frac{1}{1-\sigma^{IPB}_{h,j}}}. \tag{4.69}$$

$$I^{PB2}_{h,j,k,c,t} = \mu^{IPB2}_{h,j,k,c} \left(\frac{P^{IPB2}_{h,j,k,c,t}}{P^{IPB1}_{h,j,k,t}} \right)^{-\sigma^{IPB1}_{h,j}} I^{PB1}_{h,j,k,t}, \quad c \in \{D\}, \quad k \in \{C, P\}, \tag{4.70}$$

$$P^{IPB1}_{h,j,k,t} = \left(\sum_{c \in \{D\}} \mu^{IPB2}_{h,j,k,c} \left(P^{IPB2}_{h,j,k,c,t} \right)^{1-\sigma^{IPB1}_{h,j}} \right)^{\frac{1}{1-\sigma^{IPB1}_{h,j}}}, \quad k \in \{C, P\}, \tag{4.71}$$

where $\mu^{IPB2}_{k,j,C,D} = \mu^{IPB2}_{k,j,P,D} = 1$.[13]

[13] which yields

$$\begin{aligned} I^{PB2}_{h,j,k,D,t} &= I^{PB1}_{h,j,k,t} \\ P^{IPB1}_{h,j,k,t} &= P^{IPB2}_{h,j,k,D,t} \end{aligned}$$

$$
\begin{aligned}
P^{IB2}_{h,j,k,F,t} &= \left(1+s^{G,IB,Dwe}_{j,t}+t^{IB,VAT}_{j,t}+t^{IB,DutyV}_{j,t}+t^{IB,DutyQ}_{j,D,k,t}\right. \\
&\left.-s^{G,IB,Spe}_{j,t}-s^{EU,IB,Spe}_{j,t}\right)\left(1+t^{IB,Cus}_{j,t}\right)P^{F}_{t}, \\
k &\in \{C,P\},
\end{aligned}
\tag{4.72}
$$

$$
I^{PB3}_{h,j,k,D,i,t} = \varrho_{i,k}\left(\frac{P^{IPB3}_{h,j,k,D,i,t}}{P^{IPB2}_{h,j,k,D,t}}\right)^{-E_k} I^{PB2}_{h,j,k,D,t}, \quad i\in[1,n_k], \quad k\in\{C,P\}, \tag{4.73}
$$

$$
P^{IPB2}_{h,j,k,D,t} = \left(\int_0^{n_k}\varrho_{i,k}\left(P^{IPB3}_{h,j,k,D,i,t}\right)^{1-E_k}di\right)^{\frac{1}{1-E_k}}, \quad k\in\{C,P\}. \tag{4.74}
$$

$$
\begin{aligned}
P^{IPB3}_{h,j,k,D,i,t} &= \left(1+s^{G,IB,Dwe}_{j,t}+t^{IB,VAT}_{j,t}+t^{IB,DutyV}_{j,t}+t^{IB,DutyQ}_{j,D,k,t}\right. \\
&\left.-s^{G,IB,Spe}_{j,t}-s^{EU,IB,Spe}_{j,t}\right)P^{Y}_{i,k,t}, \\
i &\in [0,n_k], \quad k\in\{C,P\}.
\end{aligned}
\tag{4.75}
$$

4.4 Symmetric equilibrium

According to eq. (4.16) in section 4.2, each firm within an industry faces a demand curve for their product which is different from the demand curve faced by other firms of the industry only if the scale parameters, $\varrho_{h,j}$, differ among the firms of the industry. We now assume that the demand faced by each firm of an industry is the same, i.e. we set $\varrho_{h,j}$ to be equal for all h of sector j:

$$
\varrho_{h,j} = \varrho_j \quad \forall h\in[0,n_j], \quad j\in\{C,P\}.
$$

In addition we make the assumption that

$$
n_j^{\frac{1}{1-E_j}}\varrho_j^{\frac{1}{1-E_j}} = 1 \quad \forall j\in\{C,P\}.
$$

This amounts to .....

Furthermore we assume that the production techonology is the same for all firms within a sector. We may therefore omit the subscript h denoting firm on all production function parameters.

Finally, as seen from equations (4.56), (4.59), (4.64), (4.67), (4.72) and (**??**) the prices paid for goods used as inputs in production depend on the delivering and demanding sectors, but not on the specific firm using the input.

for $k\in\{C,P\}$.

When the production technology and the demand and factor prices faced by each firm of a sector is independent of the specific firm, equilibrium will be symmetric in the sense that all quantities as well as price indices defined by production technology, shadow prices and the chosen output price will be the same for all firms within a given sector.

4.5 Aggregation across firms

When the prices faced and chosen by all firms within a sector are the same, we may omit the index h denoting firm on all prices (including shadow prices), and from now on we shall be working with only sector-specific prices. In addition, we shall aggregate all quantitites across firms of an industry, thus obtaining the total quantities used or produced by the industry. Letting $X_{h,j,t}$ denote any quantity associated with firm h of sector j, we define the total quantity of the sector as

$$X_{j,t} \equiv \int_0^{n_j} X_{h,j,t} dh = n_j X_{h,j,t},$$

where the last equality follows from the fact that $X_{h,j,t}$ is the same for all $h \in [1, n_j]$.

4.5.1 Intertemporal optimization

When aggregating across firms in the set of equations constituting the solution to the individual firm's intertemporal optimization, i.e. equations (4.42) – (4.51) above, quantities appear in two ways. One is as ratios, e.g. $\frac{I^{PM}_{h,j,t}}{K^{PM}_{h,j,t-1}}$ appearing in the machinery capital installation cost function. In this case we obtain

$$\frac{I^{PM}_{h,j,t}}{K^{PM}_{h,j,t-1}} = \frac{n_j I^{PM}_{h,j,t}}{n_j K^{PM}_{h,j,t-1}} = \frac{I^{PM}_{j,t}}{K^{PM}_{j,t-1}}.$$

It should be noted that the fact that investments and capital stock are dated differently does not cause any problems because the number of firms in an industry does not vary with time.

The other way in which quantities appear are in the demand functions for inputs used in production, e.g. the demand for materials, $M_{h,j,t} = \mu^M_{h,j} \left(\frac{P^M_{h,j,t}}{P^O_{h,j,t}}\right)^{-\sigma^Y_{h,j}} Y^{Gross}_{h,j,t}$. Initially we may omit the subscript h on $\mu^M_{h,j}$ and on $\sigma^Y_{h,j}$, because production functions are the same across firms of the industry. Because prices are also the same for all firms, we omit the firm index on the prices. We are then left with:

$$\begin{aligned} M_{h,j,t} &= \mu^M_j \left(\frac{P^M_{j,t}}{P^O_{j,t}}\right)^{-\sigma^Y_j} Y^{Gross}_{h,j,t} \Rightarrow n_j M_{h,j,t} = \mu^M_j \left(\frac{P^M_{j,t}}{P^O_{j,t}}\right)^{-\sigma^Y_j} n_j Y^{Gross}_{h,j,t} \\ &\Rightarrow \ M_{j,t} = \mu^M_j \left(\frac{P^M_{j,t}}{P^O_{j,t}}\right)^{-\sigma^Y_j} Y^{Gross}_{j,t}. \end{aligned}$$

From a superficial point of view, aggregation may thus be done simply by omitting all subscripts denoting the firm. Doing this on eqs. (4.42) – (4.51) which make up the solution to the intertemporal optimization of the firms, we now obtain (in the same order as above):

$$\left(1-t_t^{Cor}\right)2k_j^{I,P,M}\frac{I_{j,t}^{PM}}{K_{j,t-1}^{PM}}\frac{P_{j,t}^{O}}{P_{j,t}^{IPM}}+1-w_{j,t}^{DP}=\frac{Q_{j,t}^{KPM}}{P_{j,t}^{IPM}}+Q_{j,t}^{KPM,Book}, \tag{4.76}$$

$$\left(1-t_t^{Cor}\right)2k_j^{I,P,B}\frac{I_{j,t}^{PB}}{K_{j,t-1}^{PB}}\frac{P_{j,t}^{O}}{P_{j,t}^{IPB}}+1-w_{j,t}^{DP}=\frac{Q_{j,t}^{KPB}}{P_{j,t}^{IPB}}+Q_{j,t}^{KPB,Book}, \tag{4.77}$$

$$\begin{aligned}&\left\{\left(1-t_t^{Cor}\right)\left[P_{j,t}^{KPM}+k_j^{I,P,M}\left(\frac{I_{j,t}^{PM}}{K_{j,t-1}^{PM}}\right)^2P_{j,t}^{O}-i_tw_{j,t-1}^{DP}P_{j,t-1}^{IPM}-t_{j,t}^{P,Weight}\right]\right.\\&\quad\left.+w_{j,t}^{DP}P_{j,t}^{IPM}\left(1-\delta_{j,t}^{PM}-\frac{P_{j,t-1}^{IPM}}{P_{j,t}^{IPM}}\right)\right\}\\&=\left(i_t+\frac{risk_t}{1-t_t^Z}\right)Q_{j,t-1}^{KPM}+\delta_{j,t}^{PM}Q_{j,t}^{KPM}-\left(Q_{j,t}^{KPM}-Q_{j,t-1}^{KPM}\right),\end{aligned} \tag{4.78}$$

$$\begin{aligned}&\left\{\left(1-t_t^{Cor}\right)\left[P_{j,t}^{KPB}+k_j^{I,P,B}\left(\frac{I_{j,t}^{PB}}{K_{j,t-1}^{PB}}\right)^2P_{j,t}^{O}-\left(i_tw_{j,t-1}^{DP}+t_{j,t}^{P,Land}\right)P_{j,t-1}^{IPB}\right]\right.\\&\quad\left.+w_{j,t}^{DP}P_{j,t}^{IPB}\left(1-\delta_{j,t}^{PB}-\frac{P_{j,t-1}^{IPB}}{P_{j,t}^{IPB}}\right)\right\}\\&=\left(i_t+\frac{risk_t}{1-t_t^Z}\right)Q_{j,t-1}^{KPB}+\delta_{j,t}^{PB}Q_{j,t}^{KPB}-\left(Q_{j,t}^{KPB}-Q_{j,t-1}^{KPB}\right),\end{aligned} \tag{4.79}$$

$$\begin{aligned}&t_t^{Cor}\delta_{j,t}^{PM,Book}\\&=\left(i_t+\frac{risk_t}{1-t_t^Z}\right)Q_{j,t-1}^{KPM,Book}+\delta_{j,t}^{PM,Book}Q_{j,t}^{KPM,Book}-\left(Q_{j,t}^{KPM,Book}-Q_{j,t-1}^{KPM,Book}\right),\end{aligned} \tag{4.80}$$

$$\begin{aligned}&t_t^{Cor}\delta_{j,t}^{PB,Book}\\&=\left(i_t+\frac{risk_t}{1-t_t^Z}\right)Q_{j,t-1}^{KPB,Book}+\delta_{j,t}^{PB,Book}Q_{j,t}^{KPB,Book}-\left(Q_{j,t}^{KPB,Book}-Q_{j,t-1}^{KPB,Book}\right),\end{aligned} \tag{4.81}$$

$$M_{j,t} = \mu_j^M \left(\frac{P_{j,t}^M}{P_{j,t}^O}\right)^{-\sigma_j^Y} Y_{j,t}^{Gross}, \tag{4.82}$$

$$H_{j,t} = \mu_j^H \left(\frac{P_{j,t}^H}{P_{j,t}^O}\right)^{-\sigma_j^Y} Y_{j,t}^{Gross}, \tag{4.83}$$

$$(1+g)^t L_{j,t}^D = \mu_j^L \left(\frac{\left(1+t_{j,t}^{Emp}+t_{j,t}^W\right)\frac{W_t}{(1+g)^t}}{P_{j,t}^H}\right)^{-\sigma_j^H} H_{j,t}, \tag{4.84}$$

$$K_{j,t}^{P,W} = \mu_j^{KP} \left(\frac{P_{j,t}^{KP,W}}{P_{j,t}^H}\right)^{-\sigma_j^H} H_{j,t}, \tag{4.85}$$

$$K_{j,t-1}^{PM} = \mu_j^{KPM} \left(\frac{P_{j,t}^{KPM}}{P_{j,t}^{KP,W}}\right)^{-\sigma_j^K} K_{j,t}^{P,W}, \tag{4.86}$$

$$K_{j,t-1}^{PB} = \mu_j^{KPB} \left(\frac{P_{j,t}^{KPB}}{P_{j,t}^{KP,W}}\right)^{-\sigma_j^K} K_{j,t}^{P,W}, \tag{4.87}$$

$$P_{j,t}^O = \left(\mu_j^M \left(P_{j,t}^M\right)^{1-\sigma_j^Y} + \mu_j^H \left(P_{j,t}^H\right)^{1-\sigma_j^Y}\right)^{\frac{1}{1-\sigma_j^Y}}, \tag{4.88}$$

$$P_{j,t}^H = \left(\mu_j^L \left(\left(1+t_{j,t}^{Emp}+t_{j,t}^W\right)\frac{W_t}{(1+g)^t}\right)^{1-\sigma_j^H} + \mu_j^{KP}\left(P_{j,t}^{KP,W}\right)^{1-\sigma_j^H}\right)^{\frac{1}{1-\sigma_j^H}}, \tag{4.89}$$

$$P_{j,t}^{KP,W} = \left(\mu_j^{KPM} \left(P_{j,t}^{KPM}\right)^{1-\sigma_j^K} + \mu_{k,j}^{KPB} \left(P_{j,t}^{KPB}\right)^{1-\sigma_j^K}\right)^{\frac{1}{1-\sigma_j^K}}, \tag{4.90}$$

$$P_{j,t}^O = \frac{E_j - 1}{E_j} P_{j,t}^Y. \tag{4.91}$$

4.5.2 Intratemporal optimization

Because all firms within a sector behave identically and choose the same prices and quantitites (both the quantities demanded and produced), we do not want to include the third nest given individual firms demand of goods from individual firms. The quantities in the lowest nest invoked in the model are therefore $I_{h,j,k,c,t}^{M2}$, $I_{h,j,k,c,t}^{PM2}$ and $I_{h,j,k,c,t}^{PB2}$. It should be noted that given the assumption $n_j^{\frac{1}{1-E_j}} \varrho_j^{\frac{1}{1-E_j}} = 1 \quad \forall j \in \{C, M\}$, we have that the indices $I_{k,j,i,D,t}^{M2}$, $I_{k,j,i,D,t}^{PM2}$ and $I_{k,j,i,D,t}^{PB2}$ of domestic goods may still be thought of as actual goods rather than indices of goods because we obtain

$$\begin{aligned}
I_{h,j,k,D,t}^{PB2} &= \left(\int_0^{n_k} \left(\varrho_{i,k}\right)^{\frac{1}{E_k}} \left(I_{h,j,i,D,i,t}^{PB3}\right)^{\frac{E_k-1}{E_k}} di\right)^{\frac{E_k}{E_k-1}} \\
&= \left(n_k \left(\varrho_{i,k}\right)^{\frac{1}{E_k}} \left(I_{h,j,k,D,i,t}^{PB3}\right)^{\frac{E_k-1}{E_k}}\right)^{\frac{E_k}{E_k-1}} \\
&= n_k^{\frac{E_k}{E_k-1}} \varrho_k^{\frac{1}{E_k-1}} I_{h,j,k,D,i,t}^{PB3} \\
&= n_k I_{h,j,k,D,i,t}^{PB3},
\end{aligned}$$

which shows $I^{PB2}_{h,j,k,D,t}$ to be just the sum of all goods demanded from individual firms in sector k. Likewise we have $I^{M2}_{h,j,k,D,t} = n_k I^{M3}_{h,j,k,D,i,t}$ and $I^{PM2}_{h,j,k,D,t} = n_k I^{PM3}_{h,j,k,D,i,t}$.

Furthermore we wish to integrate over all firms h in sector j just as it was done in the preceding section. Because all prices are identical for individual firms, we may set the price of the sector equal to the price of the individual firm and all quantities are found just by integrating over firms, which amounts to multiplying by the number of firms. Starting with materials of section 4.3.2, equations (4.52), (4.53), (4.54) and (4.55) yield:

$$M^1_{j,k,t} = \mu^{M1}_{j,k}\left(\frac{P^{M1}_{j,k,t}}{P^M_{j,t}}\right)^{-\sigma^M_j} M_{j,t}, \quad k \in \{C,P,G\}, \tag{4.92}$$

$$P^M_{j,t} = \left(\sum_{k\in\{C,P,G\}} \mu^{M1}_{j,k}\left(P^{M1}_{j,k,t}\right)^{1-\sigma^M_j}\right)^{\frac{1}{1-\sigma^M_j}}, \tag{4.93}$$

$$M^2_{j,k,c,t} = \mu^{M2}_{j,k,c}\left(\frac{P^{M2}_{j,k,c,t}}{P^{M1}_{j,k,t}}\right)^{-\sigma^{M1}_j} M^1_{j,k,t}, \quad c \in \{D,F\}, \quad k \in \{C,P,G\}, \tag{4.94}$$

$$P^{M1}_{j,k,t} = \left(\sum_{c\in\{D,F\}} \mu^{M2}_{j,k,c}\left(P^{M2}_{j,k,c,t}\right)^{1-\sigma^{M1}_j}\right)^{\frac{1}{1-\sigma^{M1}_j}}, \quad k \in \{C,P,G\}. \tag{4.95}$$

Because we disregard the third nest, we need to express $P^{M2}_{j,k,c,t} = P^{M2}_{h,j,k,c,t}$ in terms of producer prices of delivering firms. Using the definition of $P^{M2}_{h,j,k,D,t}$ in equation (4.58) together with equation (4.59) we have:

$$\begin{aligned}
P^{M2}_{h,j,k,D,t} &= \left(\int_0^{n_k} \varrho_{i,k}\left(P^{M3}_{h,j,k,D,i,t}\right)^{1-E_k} di\right)^{\frac{1}{1-E_k}} \\
&= \Bigg(n_k\varrho_k \Big(\Big(1 + t^{Res}_{j,t} - s^{G,P,Dwe}_{j,t} - s^{G,P,Res}_{j,t} - s^{EU,P,Res}_{j,t} + t^{M,VAT}_{j,t} + t^{M,DutyV}_{j,t} \\
&\quad + t^{M,DutyQ}_{j,D,k,t} - s^{GP,Spe}_{j,t} - s^{EU,P,Spe}_{j,t}\Big) P^Y_{i,k,t}\Big)^{1-E_k} \Bigg)^{\frac{1}{1-E_k}} \\
&= n_k^{\frac{1}{1-E_i}} \varrho_k^{\frac{1}{1-E_i}} \Big(\Big(1 + t^{Res}_{j,t} - s^{G,P,Dwe}_{j,t} - s^{G,P,Res}_{j,t} - s^{EU,P,Res}_{j,t} + t^{M,VAT}_{j,t} + t^{M,DutyV}_{j,t} \\
&\quad + t^{M,DutyQ}_{j,D,k,t} - s^{GP,Spe}_{j,t} - s^{EU,P,Spe}_{j,t}\Big) P^Y_{i,k,t}\Big) \\
&= \Big(1 + t^{Res}_{j,t} - s^{G,P,Dwe}_{j,t} - s^{G,P,Res}_{j,t} - s^{EU,P,Res}_{j,t} + t^{M,VAT}_{j,t} + t^{M,DutyV}_{j,t} \\
&\quad + t^{M,DutyQ}_{j,D,k,t} - s^{GP,Spe}_{j,t} - s^{EU,P,Spe}_{j,t}\Big) P^Y_{i,k,t}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad P^{M2}_{j,k,D,t} = &\Big(1 + t^{Res}_{j,t} - s^{G,P,Dwe}_{j,t} - s^{G,P,Res}_{j,t} - s^{EU,P,Res}_{j,t} + t^{M,VAT}_{j,t} + t^{M,DutyV}_{j,t} \\
&+ t^{M,DutyQ}_{j,D,k,t} - s^{GP,Spe}_{j,t} - s^{EU,P,Spe}_{j,t}\Big) P^Y_{k,t}, \\
k \in \ &\{C,P,G\},\ j \in \{C,P,G\}.
\end{aligned} \tag{4.96}$$

Finally, eq. (4.56) directly gives:

$$\begin{aligned} P^{M2}_{j,k,F,t} &= \Big(1 + t^{Res}_{j,t} - s^{G,P,Dwe}_{j,t} - s^{G,P,Res}_{j,t} - s^{EU,P,Res}_{j,t} + t^{M,VAT}_{j,t} + t^{M,DutyV}_{j,t} \\ &\quad + t^{M,DutyQ}_{j,D,k,t} - s^{GP,Spe}_{j,t} - s^{EU,P,Spe}_{j,t}\Big)\left(1 + t^{M,Cus}_{j,t}\right) P^{F}_{t}, \\ k &\in \{C,P,G\},\ j \in \{C,P,G\}. \end{aligned} \tag{4.97}$$

In exactly the same fashion equations (4.60), (4.61), (4.62) and (4.63) for machinery investments yield:

$$I^{PM1}_{j,k,t} = \mu^{IPM1}_{j,k}\left(\frac{P^{IPM1}_{j,k,t}}{P^{IPM}_{j,k,t}}\right)^{-\sigma^{IPM}_{j}} I^{PM}_{j,t}, \quad k \in \{P\}, \tag{4.98}$$

$$P^{IPM}_{j,t} = \left(\sum_{k\in\{P\}} \mu^{IPM1}_{j,k}\left(P^{IPM1}_{j,k,t}\right)^{1-\sigma^{IPM}_{j}}\right)^{\frac{1}{1-\sigma^{IPM}_{j}}}, \tag{4.99}$$

$$I^{PM2}_{j,k,c,t} = \mu^{IPM2}_{j,k,c}\left(\frac{P^{IPM2}_{j,k,c,t}}{P^{IPM1}_{j,k,t}}\right)^{-\sigma^{IPM1}_{j}} I^{PM1}_{j,k,t}, \quad c \in \{D,F\}, \quad k \in \{P\}, \tag{4.100}$$

$$P^{IPM1}_{j,k,t} = \left(\sum_{c\in\{D,F\}} \mu^{IPM2}_{j,k,c}\left(P^{IPM2}_{j,k,c,t}\right)^{1-\sigma^{IPM1}_{j}}\right)^{\frac{1}{1-\sigma^{IPM1}_{j}}}. \tag{4.101}$$

And from the definition of $P^{M2}_{h,j,k,c,t}$ in equation (4.66) together with equation (4.67) we obtain

$$\begin{aligned} P^{IPM2}_{h,j,k,D,t} &= \left(\int_0^{n_k} \varrho_{i,k}\left(P^{IPM3}_{h,j,k,D,i,t}\right)^{1-E_k} di\right)^{\frac{1}{1-E_k}} \\ &= \Bigg(n_k\varrho_k\Big(\Big(1 + t^{P,Reg}_{j,t} + t^{IM,VAT}_{j,t} + t^{IM,DutyV}_{j,t} + t^{IM,DutyQ}_{j,D,k,t} \\ &\quad - s^{G,IM,Spe}_{j,t} - s^{EU,IM,Spe}_{j,t}\Big) P^{Y}_{i,k,t}\Big)^{1-E_k} \Bigg)^{\frac{1}{1-E_k}} \\ &= n_k^{\frac{1}{1-E_k}} \varrho_k^{\frac{1}{1-E_k}} \Big(\Big(1 + t^{P,Reg}_{j,t} + t^{IM,VAT}_{j,t} + t^{IM,DutyV}_{j,t} + t^{IM,DutyQ}_{j,D,k,t} - \\ &\quad s^{G,IM,Spe}_{j,t} - s^{EU,IM,Spe}_{j,t}\Big) P^{Y}_{i,k,t}\Big) \\ &= \Big(1 + t^{P,Reg}_{j,t} + t^{IM,VAT}_{j,t} + t^{IM,DutyV}_{j,t} + t^{IM,DutyQ}_{j,D,k,t} \\ &\quad - s^{G,IM,Spe}_{j,t} - s^{EU,IM,Spe}_{j,t}\Big) P^{Y}_{i,k,t} \end{aligned}$$

$$\begin{aligned} \Rightarrow\ P^{IPM2}_{j,k,D,t} &= \Big(1 + t^{P,Reg}_{j,k,t} + t^{IM,VAT}_{j,t} + t^{IM,DutyV}_{j,t} + t^{IM,DutyQ}_{j,D,k,t} \\ &\quad - s^{G,IM,Spe}_{j,t} - s^{EU,IM,Spe}_{j,t}\Big) P^{Y}_{k,t}, \\ k &\in \{P\}. \end{aligned} \tag{4.102}$$

Finally, equation (4.64) directly gives:

$$
\begin{aligned}
P^{IPM2}_{j,k,F,t} &= \Big(1 + t^{P,Reg}_{j,t} + t^{IM,VAT}_{j,t} + t^{IM,DutyV}_{j,t} + t^{IM,DutyQ}_{j,D,k,t} \\
&\quad - s^{G,IM,Spe}_{j,t} - s^{EU,IM,Spe}_{j,t}\Big)\Big(1 + t^{IM,Cus}_{j,t}\Big) P^F_t, \\
k &\in \{P\},\ j \in \{C,P,G\}.
\end{aligned} \tag{4.103}
$$

For building investments we have, using equations (4.68), (4.69), (4.70) and (4.71):

$$
I^{PB1}_{j,k,t} = \mu^{IPB1}_{j,k}\left(\frac{P^{IPB1}_{j,k,t}}{P^{IPB}_{j,t}}\right)^{-\sigma^{IPB}_j} I^{PB}_{j,t}, \quad k \in \{C,P\}, \tag{4.104}
$$

$$
P^{IPB}_{j,t} = \left(\sum_{k\in\{C,P\}} \mu^{IPB1}_{j,k}\left(I^{PB1}_{j,k,t}\right)^{1-\sigma^{IPB}_j}\right)^{\frac{1}{1-\sigma^{IPB}_j}}, \tag{4.105}
$$

$$
I^{PB2}_{j,k,c,t} = \mu^{IPB2}_{j,k,c}\left(\frac{P^{IPB2}_{j,k,c,t}}{P^{IPB1}_{j,k,t}}\right)^{-\sigma^{IPB1}_j} I^{PB1}_{j,k,t}, \quad c \in \{D\}, \quad k \in \{C,P\}, \tag{4.106}
$$

$$
P^{IPB1}_{j,k,t} = \left(\sum_{c\in\{D\}} \mu^{IPB2}_{j,k,c}\left(P^{IPB2}_{j,k,c,t}\right)^{1-\sigma^{IPB1}_j}\right)^{\frac{1}{1-\sigma^{IPB1}_j}}, \quad k \in \{C,P\}. \tag{4.107}
$$

And using equation (4.74) together with equation (**??**) we obtain

$$
\begin{aligned}
P^{IPB2}_{h,j,k,D,t} &= \left(\int_0^{n_k} \varrho_{i,k}\left(P^{IPB3}_{h,j,k,D,i,t}\right)^{1-E_k} di\right)^{\frac{1}{1-E_k}} \\
&= \Bigg(n_k \varrho_k \Bigg(\Big(1 + s^{G,IB,Dwe}_{j,t} + t^{IB,VAT}_{j,t} + t^{IB,DutyV}_{j,t} + t^{IB,DutyQ}_{j,D,k,t} \\
&\quad - s^{G,IB,Spe}_{j,t} - s^{EU,IB,Spe}_{j,t}\Big) P^Y_{i,k,t}\Bigg)^{1-E_k} \Bigg)^{\frac{1}{1-E_k}} \\
&= n_k^{\frac{1}{1-E_k}} \varrho_k^{\frac{1}{1-E_k}} \Bigg(\Big(1 + s^{G,IB,Dwe}_{j,t} + t^{IB,VAT}_{j,t} + t^{IB,DutyV}_{j,t} + t^{IB,DutyQ}_{j,D,k,t} \\
&\quad - s^{G,IB,Spe}_{j,t} - s^{EU,IB,Spe}_{j,t}\Big) P^Y_{i,k,t}\Bigg) \\
&= \Big(1 + s^{G,IB,Dwe}_{j,t} + t^{IB,VAT}_{j,t} + t^{IB,DutyV}_{j,t} + t^{IB,DutyQ}_{j,D,k,t} \\
&\quad - s^{G,IB,Spe}_{j,t} - s^{EU,IB,Spe}_{j,t}\Big) P^Y_{i,k,t}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\ P^{IPB2}_{j,k,D,t} &= \Big(1 + s^{G,IB,Dwe}_{j,t} + t^{IB,VAT}_{j,t} + t^{IB,DutyV}_{j,t} + t^{IB,DutyQ}_{j,D,k,t} \\
&\quad - s^{G,IB,Spe}_{j,t} - s^{EU,IB,Spe}_{j,t}\Big) P^Y_{k,t}, \\
k &\in \{C,P\}.
\end{aligned} \tag{4.108}
$$

Finally, equation (4.72) directly gives:

$$
\begin{aligned}
P^{IB2}_{j,k,F,t} &= \Big(1 + s^{G,IB,Dwe}_{j,t} + t^{IB,VAT}_{j,t} + t^{IB,DutyV}_{j,t} + t^{IB,DutyQ}_{j,D,k,t} \\
&\quad - s^{G,IB,Spe}_{j,t} - s^{EU,IB,Spe}_{j,t}\Big)\Big(1 + t^{IB,Cus}_{j,t}\Big) P^F_t, \\
k &\in \{C,P\}.
\end{aligned} \tag{4.109}
$$

4.6 Government production

The government producer (we shall assume that there is only one) acts differently from the firms in the private sector in that it seeks to minimize costs associated with production in each period rather than maximizing profits intertemporally. There are no costs associated with installation of capital, and for convenience we may therefore think of the goverment producer as directly choosing capital rather than investments. The behaviour of the capital stock over time is determined by the assumption that the ratio of capital stocks to (net) production is constant, i.e.

$$\frac{K_{j,t}^{PM}}{Y_{j,t}^{Gross}} = \frac{K_{j,t-1}^{PM}}{Y_{j,t-1}^{Gross}}, \quad (4.110)$$

$$\frac{K_{j,t}^{PB}}{Y_{j,t}^{Gross}} = \frac{K_{j,t-1}^{PB}}{Y_{j,t-1}^{Gross}}. \quad (4.111)$$

This is a crude way of modelling the behaviour of policy makers, which just assures that the proportions of public production remains fairly stable. Investments are then given residually from the capital accumulation equations identical to those of the private sector, i.e. equations (4.9) and (4.10) above.

The production function is of the same sort used in private production, and gross production is therefore given as a function of the two capital stocks, materials and labour:[14]

$$Y_{j,t}^{Gross} = F\left[K_{j,t}^{PM}, K_{j,t}^{PB}, M_{j,t}, (1+g)^t L_{j,t}^D\right].$$

Letting $P_{j,t}^{KPM}$ and $P_{j,t}^{KPB}$ be the prices (or user-costs) of $K_{j,t}^{PM}$ and $K_{j,t}^{PB}$ respectively, costs are given as:

$$P_{j,t}^{KPM} K_{j,t}^{PM} + P_{j,t}^{KPB} K_{j,t}^{PB} + P_{j,t}^{M} M_{j,t} + \left(1 + t_{j,t}^{Emp} + t_{j,t}^{W}\right) W_t L_{j,t}^D.$$

We solve the cost minimization problem in two steps. First $M_{j,t}$ and $L_{j,t}^D$ are chosen optimally in order to minimize costs subject to the condition that production is given. The needed investments $I_{j,t}^{PM}$ and $I_{j,t}^{PB}$ may be found from the capital accumulation identities of the government sector, though we also derive first-order conditions for capital below. However, these are not used for determining the actual capital stock, but instead determine the prices $P_{j,t}^{KPM}$ and $P_{j,t}^{KPB}$. In the second step we solve the problem of minimizing costs associated with obtaining the indices of $M_{j,t}$, $I_{j,t}^{PM}$ and $I_{j,t}^{PB}$, which is exactly the same problem that we solved under private production, and therefore the solution is the same.

[14] Although there are no capital installation costs in government production, we shall maintain the destinction between net and gross production becuase we wish to use the same set of equations to describe production in both private and government production.

Considering step 1, the problem is:

$$\begin{array}{ll} \min & P_{j,t}^{KPM} K_{j,t}^{PM} + P_{j,t}^{KPB} K_{j,t}^{PB} + P_{j,t}^{M} M_{j,t} + \left(1 + t_{j,t}^{Emp} + t_{j,t}^{W}\right) W_t L_{j,t}^{D} \\ \text{s.t.} & Y_{j,t}^{Gross} = F\left[K_{j,t}^{PM}, K_{j,t}^{PB}, M_{j,t}, (1+g)^t L_{j,t}^{D}\right]. \end{array}$$

4.6.1 Solving the problem of step 1

We set up the Lagrangian

$$\begin{aligned} \mathcal{L} \;=\; & P_{j,t}^{KPM} K_{j,t}^{PM} + P_{j,t}^{KPB} K_{j,t}^{PB} + P_{j,t}^{M} M_{j,t} + \left(1 + t_{j,t}^{Emp} + t_{j,t}^{W}\right) W_t L_{j,t}^{D} \\ & -\lambda \left(F\left[K_{j,t}^{PM}, K_{j,t}^{PB}, M_{j,t}, (1+g)^t L_{j,t}^{D}\right] - Y_{j,t}^{Gross}\right), \end{aligned}$$

and the first order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial K_{j,t}^{PM}} &= P_{j,t}^{KPM} - \lambda \frac{\partial F}{\partial K_{j,t}^{PM}} = 0, \\ \frac{\partial \mathcal{L}}{\partial K_{j,t}^{PB}} &= P_{j,t}^{KPB} - \lambda \frac{\partial F}{\partial K_{j,t}^{PB}} = 0, \\ \frac{\partial \mathcal{L}}{\partial M_{j,t}} &= P_{j,t}^{M} - \lambda \frac{\partial F}{\partial M_{j,t}} = 0, \\ \frac{\partial \mathcal{L}}{\partial L_{j,t}^{D}} &= \left(1 + t_{j,t}^{Emp} + t_{j,t}^{W}\right) W_t - \lambda \frac{\partial F}{\partial \left((1+g)^t L_{j,t}^{D}\right)} (1+g)^t = 0. \end{aligned}$$

Rearranging and dividing through with $(1+g)^t$ in the last equation yields:

$$P_{j,t}^{KPM} = \lambda \frac{\partial F}{\partial K_{j,t}^{PM}}, \tag{4.112}$$

$$P_{j,t}^{KPB} = \lambda \frac{\partial F}{\partial K_{j,t}^{PB}}, \tag{4.113}$$

$$P_{j,t}^{M} = \lambda \frac{\partial F}{\partial M_{j,t}},$$

$$\frac{\left(1 + t_{j,t}^{Emp} + t_{j,t}^{W}\right) W_t}{(1+g)^t} = \lambda \frac{\partial F}{\partial \left((1+g)^t L_{j,t}^{D}\right)},$$

which are well-known conditions for cost minimization. Given the nest structure and the fact that the production function is CES, we have the following solution

$$M_{j,t} = \mu_j^M \left(\frac{P_{j,t}^M}{P_{j,t}^O}\right)^{-\sigma_j^Y} Y_{j,t}^{Gross}, \tag{4.114}$$

$$H_{j,t} = \mu_j^H \left(\frac{P_{j,t}^H}{P_{j,t}^O}\right)^{-\sigma_j^Y} Y_{j,t}^{Gross}, \tag{4.115}$$

$$(1+g)^t L_{j,t}^D = \mu_j^L \left(\frac{\left(1+t_{j,t}^{Emp}+t_{j,t}^W\right)\frac{W_t}{(1+g)^t}}{P_{j,t}^H}\right)^{-\sigma_j^H} H_{j,t}, \tag{4.116}$$

$$K_{j,t}^{P,W} = \mu_j^{KP} \left(\frac{P_{j,t}^{KP,W}}{P_{j,t}^H}\right)^{-\sigma_j^H} H_{j,t}, \tag{4.117}$$

$$K_{j,t-1}^{PM} = \mu_j^{KPM} \left(\frac{P_{j,t}^{KPM}}{P_{j,t}^{KP,W}}\right)^{-\sigma_j^K} K_{j,t}^{P,W}, \tag{4.118}$$

$$K_{j,t-1}^{PB} = \mu_j^{KPB} \left(\frac{P_{j,t}^{KPB}}{P_{j,t}^{KP,W}}\right)^{-\sigma_{k,j}^K} K_{j,t}^{P,W}, \tag{4.119}$$

where

$$P_{j,t}^O = \left(\mu_j^M \left(P_{j,t}^M\right)^{1-\sigma_j^Y} + \mu_j^H \left(P_{j,t}^H\right)^{1-\sigma_j^Y}\right)^{\frac{1}{1-\sigma_j^Y}}, \tag{4.120}$$

$$P_{j,t}^H = \left(\mu_j^L \left(\left(1+t_{j,t}^{Emp}+t_{j,t}^W\right)\frac{W_t}{(1+g)^t}\right)^{1-\sigma_j^H} + \mu_j^{KP}\left(P_{j,t}^{KP,W}\right)^{1-\sigma_j^H}\right)^{\frac{1}{1-\sigma_j^H}}, \tag{4.121}$$

$$P_{j,t}^{KP,W} = \left(\mu_j^{KPM} \left(P_{j,t}^{KPM}\right)^{1-\sigma_j^K} + \mu_j^{KPB} \left(P_{j,t}^{KPB}\right)^{1-\sigma_j^K}\right)^{\frac{1}{1-\sigma_j^K}}. \tag{4.122}$$

Equations (4.114) – (4.122) are identical to their private sector counterparts, eqs. (4.82) – (4.90). There is one important difference, however: Eqs. (4.118) and (4.119) do not determine demand for capital as a function of prices as is the case for private production, but instead yield prices, $P_{j,t}^{KPM}$ and $P_{j,t}^{KPB}$ given the capital stocks. The fact that $K_{j,t}^{PM}$ and $K_{j,t}^{PB}$ are determined each period in the way given by eqs. (4.110) and (4.111) is the reason why costs are minimized in each period rather than intertemporally.

4.6.2 The output price of the government sector

Because profits are not maximized for government production, it is not immediately clear how the output price should be chosen. We assume that the price is chosen so that it covers average costs associated with production:

$$P_{j,t}^{Y} = \frac{\left(1+t_{j,t}^{Emp}+t_{j,t}^{W}\right) W_t L_{j,t}^{D} + P_{j,t}^{M} M_{j,t} + t_{j,t}^{P,Weight} K_{j,t-1}^{PM} + t_{j,t}^{P,Land} P_{j,t-1}^{IPB} K_{j,t-1}^{PB}}{Y_{j,t}} + \frac{\left(1+k_{j,t}^{Depr}\right)\left[\delta_{j,t}^{PM} P_{j,t}^{IPM} K_{j,t-1}^{PM} + \delta_{j,t}^{PB} P_{j,t}^{IPB} K_{j,t-1}^{PB}\right]}{Y_{j,t}}. \quad (4.123)$$

Finally, the problems of minimizing costs associated with obtaining material inputs and investments are identical to those of private firms, and therefore the solution is given by eq. (4.92) – (4.109).

4.7 Appendix: Equations describing private and government production in growth- and inflation-corrected terms

In this section, finally all equations from the preceding sections are presented in the order that they follow in the computer program. At the same time, the equations are presented in terms adjusted for foreign inflation and exogenous productivity growth.

First we have the arbitrage condition eq. (4.2) :

$$V_{j,t} = \frac{1}{\left(1+i_{t+1}\right)\left(1-t_{t+1}^{Z}\right)+risk_{t+1}}\left(1-t_{t+1}^{Z}\right)\left[DIV_{j,t+1}+V_{j,t+1}\left(1+g\right)\left(1+g^{P}\right)\right],$$
$$j \in \{C,P\}.$$

Then dividends from firm is defined in eq. (4.7) :

$$
\begin{aligned}
DIV_{j,t} \;=\; & \left(1-t_t^{Cor}\right) \\
& \times \Big[\; P_{j,t}^{Y} Y_{j,t} + Y_{j,t}^{NorthSea} \\
& \quad - P_{j,t}^{M} M_{j,t} \\
& \quad - \left(1 + t_{j,t}^{Emp} + t_{j,t}^{W} + q_t^{LG}\right) W_t L_{j,t}^{D} \\
& \quad - i_t \frac{D_{j,t-1}^{P}}{(1+g)(1+g^P)} \\
& \quad + s_{j,t}^{EU,P,SetAside} + s_{j,t}^{EU,P,Rural} + o_{j,t}^{G,P,Cap} \\
& \quad - t_{j,t}^{P,Weight} \frac{K_{j,t-1}^{P,M}}{1+g} - t_{j,t}^{P,Land} \frac{P_{j,t-1}^{IPB}}{(1+g^P)} \frac{K_{j,t-1}^{P,B}}{(1+g)} \; \Big] \\
& - P_{j,t}^{IPM} I_{j,t}^{P,M} - P_{j,t}^{IPB} I_{j,t}^{P,B} \\
& + t_t^{Cor} \delta_{j,t}^{P,M,Book} \frac{K_{j,t-1}^{P,M,Book}}{(1+g)(1+g^P)} + t_t^{Cor} \delta_{j,t}^{P,B,Book} \frac{K_{j,t-1}^{P,B,Book}}{(1+g)(1+g^P)} \\
& + D_{j,t}^{P} - \frac{D_{j,t-1}^{P}}{(1+g)(1+g^P)} \\
& - \left(1 + t_t^{I,I,DutyV} + t_{j,D,t}^{I,I,DutyQ}\right) P_{j,t}^{Y} I_{j,t}^{P,I} \\
& - \left(1 + t_t^{I,I,DutyV} + t_{j,F,t}^{I,I,DutyQ}\right) \left(1 + t_t^{I,I,Cus}\right) P_{j,t}^{F} I_{j,t}^{F,I} \\
& - o_{j,t}^{PG,Lump,Cor} - o_{j,t}^{PG;Quasi} - o_{j,t}^{PG,LandRent}.
\end{aligned}
$$

Corporate debt is a fixed fraction of the replacement value of the cpital stock, eq. (4.8) :

$$
D_{j,t}^{P} = w_{j,t}^{DP} \left(P_{j,t}^{IPM} K_{j,t}^{PM} + P_{j,t}^{IPB} K_{j,t}^{PB} \right), \quad j \in \{C, P\}.
$$

Machinery and building capital are given in eqs. (4.9) and (4.10) :

$$
\begin{aligned}
K_{j,t}^{PM} &= \left(1 - \delta_{j,t}^{PM}\right) \frac{K_{j,t-1}^{PM}}{(1+g)} + I_{j,t}^{PM}, \quad j \in \{C, P, G\}, \\
K_{j,t}^{PB} &= \left(1 - \delta_{j,t}^{PB}\right) \frac{K_{j,t-1}^{PB}}{(1+g)} + I_{j,t}^{PB}, \quad j \in \{C, P, G\}.
\end{aligned}
$$

Net output are given by subtracting installation costs from gross produktion, eq. (4.13) :

$$
Y_{j,t} = Y_{j,t}^{Gross} - k_j^{I,P,M} \frac{\left(I_{j,t}^{PM}\right)^2}{K_{j,t-1}^{PM} \;/\; (1+g)} - k_j^{I,P,B} \frac{\left(I_{j,t}^{PB}\right)^2}{K_{j,t-1}^{PB} \;/\; (1+g)}, \quad j \in \{C, P, G\}.
$$

The book value of building and machinery capital (eqs. (4.14) and (4.15)) is :

$$
\begin{aligned}
K_{j,t}^{PM,Book} &= \left(1-\delta_{j,t}^{PM,Book}\right)\frac{K_{j,t-1}^{PM,Book}}{(1+g)\left(1+g^{P}\right)}+P_{j,t}^{IPM}I_{j,t}^{PM}, \quad j\in\{C,P\}, \\
K_{j,t}^{PB,Book} &= \left(1-\delta_{j,t}^{PB,Book}\right)\frac{K_{j,t-1}^{PB,Book}}{(1+g)\left(1+g^{P}\right)}+P_{j,t}^{IPB}I_{j,t}^{PB}, \quad j\in\{C,P\}.
\end{aligned}
$$

By definition PKPM and PKPB are the marginal revenue products of machinery and building capital (eqs. (4.32) and (4.33)):

$$
\begin{aligned}
P_{h,j,t}^{O}\frac{\partial F}{\partial K_{h,j,t-1}^{PB}} &= P_{h,j,t}^{KPB}, \\
P_{h,j,t}^{O}\frac{\partial F}{\partial K_{h,j,t-1}^{PM}} &= P_{h,j,t}^{KPM}.
\end{aligned}
$$

The equations governing the optimal investment dicisions are given by equations (4.76) and (4.77) :

$$
\begin{aligned}
\frac{Q_{j,t}^{KPM}}{P_{j,t}^{IPM}}+Q_{j,t}^{KPM,Book} &= \left[1-w_{j,t}^{DP}+\left(1-t_{t}^{Cor}\right)\frac{P_{j,t}^{O}}{P_{j,t}^{IPM}}k_{j}^{I,P,M}\cdot 2\cdot\frac{I_{j,t}^{PM}}{K_{j,t-1}^{PM}/(1+g)}\right], \\
j &\in \{C,P\}, \\
\frac{Q_{j,t}^{KPB}}{P_{j,t}^{IPB}}+Q_{j,t}^{KPB,Book} &= \left[1-w_{j,t}^{DP}+\left(1-t_{t}^{Cor}\right)\frac{P_{j,t}^{O}}{P_{j,t}^{IPB}}k_{j}^{I,P,B}\cdot 2\cdot\frac{I_{j,t}^{PB}}{K_{j,t-1}^{PB}/(1+g)}\right], \\
j &\in \{C,P\}.
\end{aligned}
$$

Determinition of shadow prices are given in equations (4.78) – (4.81), which are the the first order conditions for the stock variables of the firm. Equations (4.78) and (4.79) :

$$
\begin{aligned}
& \left(i_t + \frac{risk_t}{1-t_t^Z}\right) \frac{Q_{j,t-1}^{KPM}}{1+g^P} + \delta_{j,t}^{PM} Q_{j,t}^{KPM} - Q_{j,t}^{KPM} + \frac{Q_{j,t-1}^{KPM}}{1+g^P} \\
= & \left(1-t_t^{Cor}\right) \left[P_{j,t}^{KPM} + P_{j,t}^{O} k_j^{I,P,M} \left(\frac{I_{j,t}^{PM}}{K_{j,t-1}^{PM} / (1+g)} \right)^2 - i_t \; w_{j,t-1}^{DP} \frac{P_{j,t-1}^{IPM}}{1+g^P} - t_{j,t}^{P,Weight} \right] \\
& + w_{j,t}^{DP} \; P_{j,t}^{IPM} \left(1 - \delta_{j,t}^{PM} - \frac{P_{j,t-1}^{IPM} \; / \; (1+g^P)}{P_{j,t}^{IPM}} \right) \\
j \in & \; \{C, P\},
\end{aligned}
$$

$$
\begin{aligned}
& \left(i_t + \frac{risk_t}{1-t_t^Z}\right) \frac{Q_{j,t-1}^{KPB}}{1+g^P} + \delta_{j,t}^{PB} Q_{j,t}^{KPB} - Q_{j,t}^{KPB} + \frac{Q_{j,t-1}^{KPB}}{1+g^P} \\
= & \left(1-t_t^{Cor}\right) \left[P_{j,t}^{KPB} + P_{j,t}^{O} k_j^{I,P,B} \left(\frac{I_{j,t}^{PB}}{K_{j,t-1}^{PB} / (1+g)} \right)^2 - \left(i_t \; w_{j,t-1}^{DP} + t_{j,t}^{P,Land} \right) \frac{P_{j,t-1}^{IPB}}{1+g^P} \right] \\
& + w_{j,t}^{DP} P_{j,t}^{IPB} \left(1 - \delta_{j,t}^{PB} - \frac{P_{j,t-1}^{IPB} / (1+g^P)}{P_{j,t}^{IPB}} \right) \\
j \in & \; \{C, P\},
\end{aligned}
$$

and for the book value (4.80) and (4.81) :

$$
\begin{aligned}
& \left(i_t + \frac{risk_t}{1-t_t^Z}\right) Q_{j,t-1}^{KPM,Book} + \delta_{j,t}^{PM,Book} Q_{j,t}^{KPM,Book} - Q_{j,t}^{KPM,Book} + Q_{j,t-1}^{KPM,Book} \\
= & \; t_t^{Cor} \delta_{j,t}^{PM,Book}, \\
j \in & \; \{C, P\},
\end{aligned}
$$

$$
\begin{aligned}
& \left(i_t + \frac{risk_t}{1-t_t^Z}\right) Q_{j,t-1}^{KPB,Book} + \delta_{j,t}^{PB,Book} Q_{j,t}^{KPB,Book} - Q_{j,t}^{KPB,Book} + Q_{j,t-1}^{KPB,Book} \\
= & \; t_t^{Cor} \delta_{j,t}^{PB,Book}, \\
j \in & \; \{C, P\}.
\end{aligned}
$$

For a given gross production, the demand for the indices of materials and value added is determined in equations (4.82) and (4.83). Given these, (4.84) determine labour demand, and the last three equations ((4.85), (4.86) and (4.87)) yield the value of capital and three prices:

$$
\begin{aligned}
M_{j,t} &= \mu_j^M \left(\frac{P_{j,t}^M}{P_{j,t}^O} \right)^{-\sigma_j^Y} Y_{j,t}^{Gross}, \quad j \in \{C, P, G\}, \\
H_{j,t} &= \mu_j^H \left(\frac{P_{j,t}^H}{P_{j,t}^O} \right)^{-\sigma_j^Y} Y_{j,t}^{Gross}, \quad j \in \{C, P, G\}, \\
(1+g)^t L_{j,t}^D &= \mu_j^L \left(\frac{\left(1 + t_{j,t}^{Emp} + t_{j,t}^W\right) \frac{W_t}{(1+g)^t}}{P_{j,t}^H} \right)^{-\sigma_j^H} H_{j,t}, \quad j \in \{C, P, G\}, \\
K_{j,t}^{P,W} &= \mu_j^{KP} \left(\frac{P_{j,t}^{KP,W}}{P_{j,t}^H} \right)^{-\sigma_j^H} H_{j,t}, \quad j \in \{C, P, G\}, \\
\frac{K_{j,t-1}^{PM}}{(1+g)} &= \mu_j^{KPM} \left(\frac{P_{j,t}^{KPM}}{P_{j,t}^{KP,W}} \right)^{-\sigma_j^K} K_{j,t}^{P,W}, \quad j \in \{C, P, G\}, \\
\frac{K_{j,t-1}^{PB}}{(1+g)} &= \mu_j^{KPB} \left(\frac{P_{j,t}^{KPB}}{P_{j,t}^{KP,W}} \right)^{-\sigma_j^K} K_{j,t}^{P,W}, \quad j \in \{C, P, G\},
\end{aligned}
$$

where P^O, P^H and $P^{K,P,W}$ are defined in equations (4.88), (4.89) and (4.90) :

$$
\begin{aligned}
P_{j,t}^O &= \left(\mu_j^M \left(P_{j,t}^M\right)^{1-\sigma_j^Y} + \mu_j^H \left(P_{j,t}^H\right)^{1-\sigma_j^Y} \right)^{\frac{1}{1-\sigma_j^Y}}, \quad j \in \{C, P, G\}, \\
P_{j,t}^H &= \left(\mu_j^{KP} \left(P_{j,t}^{KP,W}\right)^{1-\sigma_j^H} + \mu_j^L \left(\left(1 + t_{j,t}^{Emp} + t_{j,t}^W\right) \frac{W_t}{(1+g)^t} \right)^{1-\sigma_j^H} \right)^{\frac{1}{1-\sigma_j^H}}, \quad j \in \{C, P, G\}, \\
P_{j,t}^{KP,W} &= \left(\mu_{k,j}^{KPB} \left(P_{j,t}^{KPB}\right)^{1-\sigma_j^K} + \mu_j^{KPM} \left(P_{j,t}^{KPM}\right)^{1-\sigma_j^K} \right)^{\frac{1}{1-\sigma_j^K}}, \quad j \in \{C, P, G\}.
\end{aligned}
$$

The relation between the optimization price and the chosen product price of the firm is given by equation (4.91) :

$$
P_{j,t}^O = \frac{E_j - 1}{E_j} P_{j,t}^Y, \quad j \in \{C, P\}.
$$

We now look at the intertemporal optimization. We here have three groups, starting with materials equations (4.92), (4.93), (4.94), (4.95), (4.96) and (4.97) are:

$$
\begin{aligned}
M^1_{j,k,t} &= \mu^{M1}_{j,k}\left(\frac{P^{M1}_{j,k,t}}{P^M_{j,t}}\right)^{-\sigma^M_j} M_{j,t}, \quad k\in\{C,P,G\}, \quad j\in\{C,P,G\},\\
P^M_{j,t} &= \left(\sum_{k\in\{C,P,G\}}\mu^{M1}_{j,k}\left(P^{M1}_{j,k,t}\right)^{1-\sigma^M_j}\right)^{\frac{1}{1-\sigma^M_j}}, \quad j\in\{C,P,G\},\\
M^2_{j,k,c,t} &= \mu^{M2}_{j,k,c}\left(\frac{P^{M2}_{j,k,c,t}}{P^{M1}_{j,k,t}}\right)^{-\sigma^{M1}_j} M^1_{j,k,t}, \quad c\in\{D,F\}, \quad k\in\{C,P,G\}, \quad j\in\{C,P,G\},\\
P^{M1}_{j,k,t} &= \left(\sum_{c\in\{D,F\}}\mu^{M2}_{j,k,c}\left(P^{M2}_{j,k,c,t}\right)^{1-\sigma^{M1}_j}\right)^{\frac{1}{1-\sigma^{M1}_j}}, \quad k\in\{C,P,G\}, \quad j\in\{C,P,G\},\\
P^{M2}_{j,k,D,t} &= \left(1+t^{Res}_{j,t}-s^{G,P,Dwe}_{j,t}-s^{G,P,Res}_{j,t}-s^{EU,P,Res}_{j,t}+t^{M,VAT}_{j,t}+t^{M,DutyV}_{j,t}\right.\\
&\quad \left.+t^{M,DutyQ}_{j,D,k,t}-s^{GP,Spe}_{j,t}-s^{EU,P,Spe}_{j,t}\right)P^Y_{k,t},\\
k &\in \{C,P,G\},\ j\in\{C,P,G\},\\
P^{M2}_{j,k,F,t} &= \left(1+t^{Res}_{j,t}-s^{G,P,Dwe}_{j,t}-s^{G,P,Res}_{j,t}-s^{EU,P,Res}_{j,t}+t^{M,VAT}_{j,t}+t^{M,DutyV}_{j,t}\right.\\
&\quad \left.+t^{M,DutyQ}_{j,D,k,t}-s^{GP,Spe}_{j,t}-s^{EU,P,Spe}_{j,t}\right)\left(1+t^{M,Cus}_{j,t}\right)P^F_t,\\
k &\in \{C,P,G\},\ j\in\{C,P,G\},
\end{aligned}
$$

for machinary investments the equations are (4.98), (4.99), (4.100), (4.101), (4.102) and (4.103) :

$$
\begin{aligned}
I^{PM1}_{j,k,t} &= \mu^{IPM1}_{j,k}\left(\frac{P^{IPM1}_{j,k,t}}{P^{IPM}_{j,k,t}}\right)^{-\sigma^{IPM}_j} I^{PM}_{j,t}, \quad k\in\{P\}, \quad j\in\{C,P,G\},\\
P^{IPM}_{j,t} &= \left(\sum_{k\in\{P\}}\mu^{IPM1}_{j,k}\left(P^{IPM1}_{j,k,t}\right)^{1-\sigma^{IPM}_j}\right)^{\frac{1}{1-\sigma^{IPM}_j}}, \quad j\in\{C,P,G\},\\
I^{PM2}_{j,k,c,t} &= \mu^{IPM2}_{j,k,c}\left(\frac{P^{IPM2}_{j,k,c,t}}{P^{IPM1}_{j,k,t}}\right)^{-\sigma^{IPM1}_j} I^{PM1}_{j,k,t}, \quad c\in\{D,F\}, \quad k\in\{P\}, \quad j\in\{C,P,G\},\\
P^{IPM1}_{j,k,t} &= \left(\sum_{c\in\{D,F\}}\mu^{IPM2}_{j,k,c}\left(P^{IPM2}_{j,k,c,t}\right)^{1-\sigma^{IPM1}_j}\right)^{\frac{1}{1-\sigma^{IPM1}_j}}, \quad j\in\{C,P,G\},\\
P^{IPM2}_{j,k,D,t} &= \left(1+t^{P,Reg}_{j,t}+t^{IM,VAT}_{j,t}+t^{IM,DutyV}_{j,t}+t^{IM,DutyQ}_{j,D,k,t}\right.\\
&\quad \left.-s^{G,IM,Spe}_{j,t}-s^{EU,IM,Spe}_{j,t}\right)P^{Y}_{k,t},\\
k &\in \{P\},\ j\in\{C,P,G\},\\
P^{IPM2}_{j,k,F,t} &= \left(1+t^{P,Reg}_{j,t}+t^{IM,VAT}_{j,t}+t^{IM,DutyV}_{j,t}+t^{IM,DutyQ}_{j,D,k,t}\right.\\
&\quad \left.-s^{G,IM,Spe}_{j,t}-s^{EU,IM,Spe}_{j,t}\right)\left(1+t^{IM,Cus}_{j,t}\right)P^{F}_{t},\\
k &\in \{P\},\ j\in\{C,P,G\},
\end{aligned}
$$

and finally we have the six equations for building investment which are (4.104), (4.105), (4.106), (4.107), (4.108) and (4.109) :

$$
\begin{aligned}
I_{j,k,t}^{PB1} &= \mu_{j,k}^{IPB1}\left(\frac{P_{j,k,t}^{IPB1}}{P_{j,t}^{IPB}}\right)^{-\sigma_j^{IPB}} I_{j,t}^{PB}, \quad k\in\{C,P\}, \quad j\in\{C,P,G\},\\
P_{j,t}^{IPB} &= \left(\sum_{k\in\{C,P\}}\mu_{j,k}^{IPB1}\left(P_{j,k,t}^{IPB1}\right)^{1-\sigma_j^{IPB}}\right)^{\frac{1}{1-\sigma_j^{IPB}}}, \quad j\in\{C,P,G\},\\
I_{j,k,c,t}^{PB2} &= \mu_{j,k,c}^{IPB2}\left(\frac{P_{j,k,c,t}^{IPB2}}{P_{j,k,t}^{IPB1}}\right)^{-\sigma_j^{IPB1}} I_{j,k,t}^{PB1}, \quad c\in\{D\}, \quad k\in\{C,P\}, \quad j\in\{C,P,G\},\\
P_{j,k,t}^{IPB1} &= \left(\sum_{c\in\{D\}}\mu_{j,k,c}^{IPB2}\left(P_{j,k,c,t}^{IPB2}\right)^{1-\sigma_j^{IPB1}}\right)^{\frac{1}{1-\sigma_j^{IPB1}}}, \quad k\in\{C,P\}, \quad j\in\{C,P,G\},\\
P_{j,k,D,t}^{IPB2} &= \left(1+s_{j,t}^{G,IB,Dwe}+t_{j,t}^{IB,VAT}+t_{j,t}^{IB,DutyV}+t_{j,D,k,t}^{IB,DutyQ}\right.\\
&\left.-s_{j,t}^{G,IB,Spe}-s_{j,t}^{EU,IB,Spe}\right)P_{k,t}^{Y},\\
k &\in \{C,P\},\\
P_{j,k,F,t}^{IPB2} &= \left(1+s_{j,t}^{G,IB,Dwe}+t_{j,t}^{IB,VAT}+t_{j,t}^{IB,DutyV}+t_{j,D,k,t}^{IB,DutyQ}\right.\\
&\left.-s_{j,t}^{G,IB,Spe}-s_{j,t}^{EU,IB,Spe}\right)\left(1+t_{j,t}^{IB,Cus}\right)P_{t}^{F},\\
k &\in \{C,P\}.
\end{aligned}
$$

We now shifts to the government producer and here we have that the behaviour of the capital stock over time is determined by the assumption that the ratio of capital stocks to production is constant, which is the result of equations (4.110) and (4.111) :

$$
\begin{aligned}
\frac{K_{j,t}^{PM}}{Y_{j,t}^{Gross}} &= \frac{K_{j,t-1}^{PM}}{Y_{j,t-1}^{Gross}}, \quad j\in\{G\},\\
\frac{K_{j,t}^{PB}}{Y_{j,t}^{Gross}} &= \frac{K_{j,t-1}^{PB}}{Y_{j,t-1}^{Gross}}, \quad j\in\{G\}.
\end{aligned}
$$

For the government sector equations (4.114), (4.115), (4.116), (4.117), (4.118), (4.119), (4.120), (4.121) and (4.122) are identical to their private sector counterparts (eqs. (4.82) – (4.90)) so the equations above represent also the equations for the government sector.

We assume that the price is chosen so that it covers average cost associated with production, which is the result of equation (4.123) :

$$
\begin{aligned}
P_{j,t}^{Y} &= \frac{\left(1+t_{j,t}^{Emp}+t_{j,t}^{W}\right)W_t L_{j,t}^{D}+P_{j,t}^{M}M_{j,t}+t_{j,t}^{P,Weight}\frac{K_{j,t-1}^{PM}}{(1+g)}+t_{j,t}^{P,Land}\frac{P_{j,t-1}^{IPB}K_{j,t-1}^{PB}}{(1+g)(1+g^{P})}}{Y_{j,t}}\\
&+\frac{\left(1+k_{j,t}^{Depr}\right)\left[P_{j,t}^{IPM}\cdot\delta_{j,t}^{PM}\frac{K_{j,t-1}^{PM}}{(1+g)}+P_{j,t}^{IPB}\cdot\delta_{j,t}^{PB}\frac{K_{j,t-1}^{PB}}{(1+g)}\right]}{Y_{j,t}}.
\end{aligned}
$$

Finally, the problems of minimizing cost associated with obtaining material inputs and investments are identical to those of private firms, and therefore the solution is the same and given by equations (4.92) - (4.109). So the equations above also represent the equations for the government sector.

4.8 Appendix: Production technology

In this section, all production functions of the various nests are presented for the convenience of the reader:

$$
\begin{aligned}
Y^{Gross}_{h,j,t} &= \left(\left(\mu^M_{h,j}\right)^{\frac{1}{\sigma^Y_{h,j}}} \left(M_{h,j,t}\right)^{\frac{\sigma^Y_{h,j}-1}{\sigma^Y_{h,j}}} + \left(\mu^H_{h,j}\right)^{\frac{1}{\sigma^Y_{h,j}}} \left(H_{h,j,t}\right)^{\frac{\sigma^Y_{h,j}-1}{\sigma^Y_{h,j}}} \right)^{\frac{\sigma^Y_{h,j}}{\sigma^Y_{h,j}-1}} \\
M_{h,j,t} &= \left(\sum_{k\in\{C,P,G\}} \left(\mu^{M1}_{h,j,k}\right)^{\frac{1}{\sigma^M_{h,j}}} \left(M^1_{h,j,k,t}\right)^{\frac{\sigma^M_{h,j}-1}{\sigma^M_{h,j}}} \right)^{\frac{\sigma^M_{h,j}}{\sigma^M_{h,j}-1}} \\
H_{h,j,t} &= \left(\left(\mu^L_{h,j}\right)^{\frac{1}{\sigma^H_{h,j}}} \left((1+g)^t L^D_{h,j,t}\right)^{\frac{\sigma^H_{h,j}-1}{\sigma^H_{h,j}}} + \left(\mu^{KP}_{h,j}\right)^{\frac{1}{\sigma^H_{h,j}}} \left(K^{P,W}_{h,j,t}\right)^{\frac{\sigma^H_{h,j}-1}{\sigma^H_{h,j}}} \right)^{\frac{\sigma^H_{h,j}}{\sigma^H_{h,j}-1}} \\
K^{P,W}_{h,j,t} &= \left(\left(\mu^{KPM}_{h,j}\right)^{\frac{1}{\sigma^K_{h,j}}} \left(K^{PM,W}_{h,j,t}\right)^{\frac{\sigma^K_{h,j}-1}{\sigma^K_{h,j}}} + \left(\mu^{KPB}_{h,j}\right)^{\frac{1}{\sigma^K_{h,j}}} \left(K^{PB,W}_{h,j,t}\right)^{\frac{\sigma^K_{h,j}-1}{\sigma^K_{h,j}}} \right)^{\frac{\sigma^K_{h,j}}{\sigma^K_{h,j}-1}} .
\end{aligned}
$$

Note vedrørende følgende[15]

$$
M^1_{h,j,k,t} = \left(\left(\mu^{M2}_{h,j,k,D}\right)^{\frac{1}{\sigma^{M1}_{h,j}}} \left(M^2_{h,j,k,D,t}\right)^{\frac{\sigma^{M1}_{h,j}-1}{\sigma^{M1}_{h,j}}} + \left(\mu^{M2}_{h,j,k,F}\right)^{\frac{1}{\sigma^{M1}_{h,j}}} \left(M^2_{h,j,k,F,t}\right)^{\frac{\sigma^{M1}_{h,j}-1}{\sigma^{M1}_{h,j}}} \right)^{\frac{\sigma^{M1}_{h,j}}{\sigma^{M1}_{h,j}-1}}
$$

where $\mu^{M2}_{h,j,C,D} = \mu^{M2}_{h,j,G,D} = 1,\ \mu^{M2}_{h,j,C,F} = \mu^{M2}_{h,j,G,F} = 0,$

$$
M^2_{h,j,k,D,t} = \left(\int_0^{n_k} \left(\varrho_{i,k}\right)^{\frac{1}{E_k}} \left(M^3_{h,j,k,i,D,t}\right)^{\frac{E_k-1}{E_k}} di \right)^{\frac{E_k}{E_k-1}} ,
$$

[15] This amounts to

$$
\begin{aligned}
M^1_{h,j,P,t} &= \left(\left(\mu^{M2}_{h,j,P,D}\right)^{\frac{1}{\sigma^{M1}_{h,j}}} \left(M^2_{h,j,P,D,t}\right)^{\frac{\sigma^{M1}_{h,j}-1}{\sigma^{M1}_{h,j}}} + \left(\mu^{M2}_{h,j,P,F}\right)^{\frac{1}{\sigma^{M1}_{h,j}}} \left(M^2_{h,j,P,F,t}\right)^{\frac{\sigma^{M1}_{h,j}-1}{\sigma^{M1}_{h,j}}} \right)^{\frac{\sigma^{M1}_{h,j}}{\sigma^{M1}_{h,j}-1}} \\
M^1_{h,j,C,t} &= M^2_{h,j,C,D,t} \\
M^1_{h,j,G,t} &= M^2_{h,j,G,D,t}
\end{aligned}
$$

$$I_{h,j,t}^{PM} = \left(\sum_{k \in \{P\}} \left(\mu_{h,j,k}^{IPM1}\right)^{\frac{1}{\sigma_{h,j}^{IPM}}} \left(I_{h,j,k,t}^{PM1}\right)^{\frac{\sigma_{h,j}^{IPM}-1}{\sigma_{h,j}^{IPM}}} \right)^{\frac{\sigma_{h,j}^{IPM}}{\sigma_{h,j}^{IPM}-1}},$$

$$I_{h,j,k,t}^{PM1} = \left(\sum_{c \in \{D,F\}} \left(\mu_{h,j,k,c}^{IPM2}\right)^{\frac{1}{\sigma_{h,j}^{IPM1}}} \left(I_{h,j,k,c,t}^{PM2}\right)^{\frac{\sigma_{h,j}^{IPM1}-1}{\sigma_{h,j}^{IPM1}}} \right)^{\frac{\sigma_{h,j}^{IPM1}}{\sigma_{h,j}^{IPM1}-1}},$$

$$I_{h,j,P,D,t}^{PM2} = \left(\int_0^{n_P} \left(\varrho_{i,P}\right)^{\frac{1}{E_P}} \left(I_{h,j,P,D,i,t}^{PM3}\right)^{\frac{E_P-1}{E_P}} di \right)^{\frac{E_P}{E_P-1}}.$$